Math Diversion Problem 406

P. Reany

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Emmy Noether introduced the notion of a representation
space—a vector space upon which the elements of
the algebra operate as linear transformations,
the composition of the linear transformations
reflecting the multiplication in the algebra.
By doing so she enables us to use
our geometric intuition.
— Emil Artin

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0IpU9Ly_6lw
Title: An Equation With Absolute Value | Problem 478
 Presenter: aplusbi
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1 The Problem

Given the relation

$$|z| + iz = z_0 = 4 + 2i\,, \tag{1}$$

find the values of z.

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2 The Solution

The first thing to say about this problem is that we should be able to solve for both components of z, even though we are given only one equation that contains z. The reason for this is because Eq. (1) has to be satisfied for both its real and complex components.

Now, I like to approach these problems by taking the complex conjugate of the Given equation, in this case yielding,

$$|z| - i\overline{z} = \overline{z}_0\,, \tag{2}$$

On setting $|z| = r$, we get

$$r + iz = z_0\,, \tag{3a}$$
$$r - i\overline{z} = \overline{z}_0\,. \tag{3b}$$

Note: Soon I'll be making replacements: $z + \overline{z} = 2a$, $z - \overline{z} = 2bi$, and $z\overline{z} = r^2$.

Now, on adding (3a) and (3b), we have that

$$2r + i(z - \overline{z}) = z_0 + \overline{z}_0\,, \tag{4}$$

or

$$2r + i(2bi) = 2a\,, \tag{5}$$

which simplified to

$$r - b = a\,. \tag{6}$$

Next, we subtract (3a) and (3b), to get

$$i(z + \overline{z}) = z_0 - \overline{z}_0\,, \tag{7}$$

or

$$a = b_0\,, \tag{8}$$

where b_0 is known because it was given to us. Now, Eqs. (3a) and (3b) can be rewritten into the forms

$$iz = z_0 - r\,, \tag{9a}$$
$$-i\overline{z} = \overline{z}_0 - r\,. \tag{9b}$$

On multiplying these together, we have that

$$z\overline{z} = (z_0 - r)(\overline{z}_0 - r)\,, \tag{10}$$

which will boil down to

$$r = \frac{r_0^2}{2a_0}\,. \tag{11}$$

So, by combining (6), (8) and (11) and using the values for a_0 and b_0 we got in Eq. (1), we eventually get

$$z = 2 + -\tfrac{3}{2}\,i\,. \tag{12}$$