Math Diversion Problem 407

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> Abstract algebra, as a conscious discipline, starts with
> Noether's 1921 paper "Ideal Theory in Rings."
> — Saunders MacLane

The YouTube video is found at:

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Source: ---
Title: ---
 Presenter: Patrick
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1 The Problem

Given the relation

$$\ln x = e^x \,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A result we'll need:

$W_0(-1) \approx -0.31813 + 1.33723i$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{5}$$

3 The Solution

The simplest way to proceed is, perhaps, just to multiply (1) through by x:

$$x \ln x = x e^x \,. \tag{6}$$

Now we take the Lambert W function across this equation, yielding

$$\ln x = x \,. \tag{7}$$

Raising e to this equation, we have that

$$x = e^x \,, \tag{8}$$

which can be rewritten as

$$x e^{-x} = 1 \,. \tag{9}$$

Multiplicity through by -1, we have that

$$-x e^{-x} = -1 \,. \tag{10}$$

Taking the Lambert W function across this equation, gives

$$-x = W(-1) \,. \tag{11}$$

Then

$$x = -W(-1) \approx 0.31813 - 1.33723 i \,. \tag{12}$$

But is that all? It seems not, at least according to WolframAlpha, which gives not only (12) but also its complex conjugate, but I have no idea how it got this second solution.