# Math Diversion Problem 411

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> Self-education is, I firmly believe, the only
> kind of education there is.
> — Isaac Asimov

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=jS6u-l55QlE
Title: A Nice Math Olympiad Exponential Equation 36^x = 3/x
 Presenter: MrMath
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## 1 The Problem

Given the relation

$$36^x = \frac{3}{x}\,, \tag{1}$$

find the values of $x$.

## 2 The Preparation

I intend to use the Lambert $W$ function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on $B$ that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert $W$ function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert $W$ function when $a = e$.

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## 3 The Solution

First, we rewrite the Given relation to

$$x36^x = 3\,, \tag{7}$$

and immediately take the Lambert $W$ function base 36 across this equation, to get

$$x = W_{36}(3) = \frac{W_n(3 \cdot \ln 36)}{\ln 36}\,. \tag{8}$$

The one real solution to $x$ is provided this way:

$$x = \frac{W_0(3 \cdot \ln 36)}{\ln 36} = \frac{W_0(3 \cdot \ln 6^2)}{2 \ln 6} = \frac{1}{2}\frac{W_0(6 \cdot \ln 6)}{\ln 6} = \frac{1}{2}\frac{\ln 6}{\ln 6} = \frac{1}{2}\,, \tag{9}$$

where $W_0(\cdot) = W(\cdot)$ is the principal value of $W$.