Math Diversion Problem 412

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People often overlook the obvious.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/V_NQx8uRDqM
Title: 9th grade exponent problem
 Presenter: MindSphereYT
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1 The Problem

Given the relation

$$x^x = \frac{7^{7^7}}{7^x}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y\ln y = B\,, \tag{4}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{5}$$

3 The Solution

First, we rewrite the Given relation to

$$(7x)^x = 7^{7^7}\,, \tag{6}$$

and then we raise both sides to the 7th power:

$$(7x)^{7x} = \left(7^{7^7}\right)^7\,. \tag{7}$$

On taking the natural logarithm of this, we get

$$7x\ln(7x) = 7\ln\left(7^{7^7}\right) = 7^7\ln\left(7^7\right)\,. \tag{8}$$

After taking the Lambert W function across this equation, we get

$$\ln(7x) = \ln\,7^7\,, \tag{9}$$

from which we get

$$7x = 7^7\,. \tag{10}$$

Therefore,

$$x = 7^6\,. \tag{11}$$