Math Diversion Problem 413

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Average talent, plus hard work and dedication,
will always beat talent by itself.
— Clinton Anderson

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=z_BdVV5MTKg
Title: Solving a 'Harvard' University entrance exams
 Presenter: The Map of Mathematics
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1 The Problem

Given the relation

$$27^x = -x\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{7}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{8}$$

3 The Solution

We begin by taking the 27^x to the other side:

$$1 = -x 27^{-x}\,, \tag{9}$$

Now we take the Lambert W function base 27 across this equation, to get

$$-x = W_{27}(1) = \frac{W(1 \cdot \ln 27)}{\ln 27} = \frac{W(3 \ln 3)}{3 \ln 3} = \frac{\ln 3}{3 \ln 3} = \frac{1}{3}\,. \tag{10}$$

Hence,

$$x = -\frac{1}{3}\,. \tag{11}$$