Math Diversion Problem 418

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February 19, 2025

It's not that I'm so smart; it's just that
I stay with problems longer.
— Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=_eiiL4r6CHQ
Title: Can you Solve Oxford University Admission Test ?
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^2 = (5 - \sqrt{24}\,)^x \,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B \,, \tag{4}$$

then

$$z = W_a(B) \,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{7}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{8}$$

3 The Solution

I'd like to start by taking the square root across (1):

$$x = \pm \left[\left(5 - \sqrt{24}\right)^{1/2} \right]^{x}\,, \tag{9}$$

and then write that in canonical form as[1]

$$-x \left[\left(5 - \sqrt{24}\right)^{1/2} \right]^{-x} = \pm 1\,. \tag{10}$$

Then

$$-x = W_{\left(5-\sqrt{24}\right)^{1/2}}(\pm 1) = \frac{W(\pm 1 \cdot \ln \left(5 - \sqrt{24}\right)^{1/2})}{\ln \left(5 - \sqrt{24}\right)^{1/2}}\,. \tag{11}$$

Hence,

$$x = \begin{cases} -2\,\dfrac{W(+\frac{1}{2}\ln(5-\sqrt{24}\,))}{\ln(5-\sqrt{24}\,)}\,, \\ -\,2\,\dfrac{W(-\frac{1}{2}\ln(5-\sqrt{24}\,))}{\ln(5-\sqrt{24}\,)}\,. \end{cases} \tag{12}$$

Now, WolframAlpha claims that (12, bottom line) is the only real-valued solution, and that the solutions I gave should be replaced by the more general solutions:

$$x = \begin{cases} -2\,\dfrac{W_n(+\frac{1}{2}\ln(5-\sqrt{24}\,))}{\ln(5-\sqrt{24}\,)}\,, \\ -\,2\,\dfrac{W_n(-\frac{1}{2}\ln(5-\sqrt{24}\,))}{\ln(5-\sqrt{24}\,)}\,. \end{cases} \tag{13}$$

[1] By 'canonical form' in this context, I mean a form that I can apply either the Lambert definition or one of the Lambert lemmas.