Math Diversion Problem 419

P. Reany

February 19, 2025

Chance favors the prepared mind.
— Louis Pasteur

This time we engage ourselves in a lemma about the Lambert W function.

1 The Problem

Prove that

$$W(x^{x+1}\ln x) = x\ln x\,. \tag{1}$$

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{7}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{8}$$

3 The Proof

Let

$$\phi = x^x \,. \tag{9}$$

Then

$$\ln \phi = x \ln x \,. \tag{10}$$

Now, multiply these last two equations together, to get

$$\phi \ln \phi = x^{x+1} \ln x \,. \tag{11}$$

Next, take the Lambert W function across this equation:

$$W(\phi \ln \phi) = \ln \phi = W(x^{x+1} \ln x) \,. \tag{12}$$

Lastly, using (9) in (12), we have that

$$W(x^{x+1} \ln x) = x \ln x \,. \tag{13}$$