Math Diversion Problem 420

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I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=p2zO7g0TeO0
Title: Harvard University Exponential Problem
 Presenter: Super Academy
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1 The Problem

Given the relation

$$4^{x^2} = x^{128}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{7}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{8}$$

3 The Solution

Let's restate:

$$4^{x^2} = x^{128} = (x^2)^{64}\,, \tag{9}$$

and then take the 64th roots on both sides:

$$(4^{1/64})^{x^2} = x^2\,. \tag{10}$$

Next, we rewrite into canonical form:

$$-x^2(4^{1/64})^{-x^2} = -1\,. \tag{11}$$

Now we can take the Lambert W function base $4^{1/64}$:

$$-x^2 = W_{4^{1/64}}(-1) = \frac{W(-1 \cdot \ln 4^{1/64})}{\ln 4^{1/64}} \tag{12a}$$

$$= \frac{W(-\frac{1}{64}\ln 4)}{(1/64)\ln 4} = 32\frac{W_n(\frac{1}{64}\ln\frac{1}{4})}{\ln 2} \tag{12b}$$

$$= 32\frac{W_n(\frac{1}{32}\ln\frac{1}{2})}{\ln 2} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{12c}$$

Therefore, for x:

$$x = \pm 4i\sqrt{2\frac{W_n(\frac{1}{32}\ln\frac{1}{2})}{\ln 2}} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{13}$$