Math Diversion Problem 424

P. Reany

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All my life I kept running into smart people.... In school there were lots of smarter kids. And when I first joined the force, sir, they had some very clever people there. And I could tell right away that it wasn't going to be easy making detective as long as they were around. What I figured that... if I worked harder than they did. Put in more time. Read the books. Kept my eyes open. Maybe I could make it happen. And I did!

— Lt. Columbo to his prisoner
(from the TV show *Columbo*)
("The Bye Bye Sky High)
IQ Murder Case")

This time we engage ourselves in a lemma the theory of complex numbers.

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=2UJtaWCA_xE
Title: Simplify A Trigonometric Expression | Problem 193
 Presenter: aplusbi
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1 The Problem

Simplify

$$\phi = \sin z\, \cos w + \sin w\, \cos z\,. \tag{1}$$

2 The Simplification

Let us write down the identity:

$$e^{i(z+w)} = e^{iz}\, e^{iw}\,. \tag{2}$$

On expanding both sides, we get

$$\cos(z+w) + i\sin(z+w) = (\cos(z) + i\sin(z))\,(\cos(w) + i\sin(w)) \tag{3a}$$

$$= \cos(z)\cos(w) - \sin(z)\sin(w) + i(\cos(z)\sin(w) + \sin(z)\cos(w))\,. \tag{3b}$$

So, now we set real parts equal to real parts and imaginary parts equal to imaginary parts. Thus,

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)\,, \tag{4a}$$

$$\sin(z+w) = \cos(z)\sin(w) + \sin(z)\cos(w)\,. \tag{4b}$$

Therefore,

$$\phi = \sin(z+w)\,. \tag{5}$$