Math Diversion Problem 428

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> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another.
> — Keith Devlin

This time we engage ourselves in a lemma the theory of complex numbers.

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=iv900JA6eiU
Title: Solving a Sextic Equation
 Presenter:  Dr. Barker
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1 The Problem

Given the relation

$$(1-x)^6 = 64x^6\,. \tag{1}$$

find the values of x.

2 The Preparation

We will need to know how to convert the six roots of unity in the complex numbers. Unity, being $1 = e^{2\pi i}$, then for unique roots:

$$1^{1/6} = e^{2\pi ik/6} \quad \text{for} \quad k \in [\,0,1,2,3,4,5\,]\,. \tag{2}$$

$$k=0: \quad e^{2\pi i0/6} = 1\,, \tag{3a}$$
$$k=1: \quad e^{2\pi i1/6} = \tfrac{1}{2}(1+\sqrt{3}i)\,, \tag{3b}$$
$$k=2: \quad e^{2\pi i2/6} = \tfrac{1}{2}(-1+\sqrt{3}i)\,, \tag{3c}$$
$$k=3: \quad e^{2\pi i3/6} = -1\,, \tag{3d}$$
$$k=4: \quad e^{2\pi i4/6} = \tfrac{1}{2}(-1-\sqrt{3}i)\,, \tag{3e}$$
$$k=5: \quad e^{2\pi i5/6} = \tfrac{1}{2}(1-\sqrt{3}i)\,. \tag{3f}$$

3 The Solution

On taking the sixth root across (1), we have that

$$1 - x = 2xe^{2\pi ik/6} \quad \text{for} \quad k \in [\,0, 1, 2, 3, 4, 5\,]\,. \tag{4}$$

Solving for x, we get

$$x = \frac{1}{1 + 2e^{2\pi ik/6}} \quad \text{for} \quad k \in [\,0, 1, 2, 3, 4, 5\,]\,. \tag{5}$$

Hence,

$$x = \begin{cases} k = 0: & 1/3\,, \\ k = 1: & (2 - \sqrt{3}i)/7\,, \\ k = 2: & -i/\sqrt{3}\,, \\ k = 3: & -1\,, \\ k = 4: & i/\sqrt{3}\,, \\ k = 5: & (2 + \sqrt{3}i)/7\,. \end{cases} \tag{6}$$