Math Diversion Problem 429: Partial Derivatives Using Chain Rule Solved by SD

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February 28, 2025

Abstract

Structured Differentiation (SD) is used to solve a problem I found on a physics forum since (2016).

> To err is human; to really foul things up
> requires partial differentiation.
> — adaptation of a quote from
> Paul Ehrlich

1 Introduction

This problem is from March 24, 2016.

`https://www.physicsforums.com/threads/partial-derivatives-using-chain-rule.863640/`

The problem statement:
Suppose that

$$\omega = g(u, v)\,, \tag{1}$$

where

$$u = x/y \quad \text{and} \quad v = z/y\,. \tag{2}$$

Using the chain rule, evaluate

$$x\,\frac{\partial \omega}{\partial x} + y\,\frac{\partial \omega}{\partial x} + z\,\frac{\partial \omega}{\partial z}\,. \tag{3}$$

Note: In SD a partial derivative is an explicit derivative.

2 Solution

In SD, we write down a functional dependence and then take a total derivative:

$$\omega = g(\mathbf{u}(\mathbf{x})) \quad \text{where} \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\,, \tag{4}$$

then we totally differentiate ω:

$$\frac{\delta\omega}{\delta\mathbf{x}} = \frac{\partial\omega}{\partial\mathbf{x}} + \frac{\partial g}{\partial\mathbf{u}}\frac{\delta\mathbf{u}}{\delta\mathbf{x}}\,. \tag{5}$$

However, we can drop the first term on the RHS as zero because ω is not explicitly dependent on $\mathbf{x}$. Furthermore, since $\mathbf{u}$ is dependent on $\mathbf{x}$ only explicitly then $\delta\mathbf{u}/\delta\mathbf{x} \to \partial\mathbf{u}/\partial\mathbf{x}$, leaving us with

$$\frac{\delta\omega}{\delta\mathbf{x}} = \frac{\partial g}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\,. \tag{6}$$

Now, the LHS term is a 1×3 matrix, and the factors on the RHS are first a 1×2 matrix followed by a 2×3 matrix. Let's write them out:

$$\left[\frac{\delta\omega}{\delta x}, \frac{\delta\omega}{\delta y}, \frac{\delta\omega}{\delta z}\right] = \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right] \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{bmatrix}\,. \tag{7}$$

Now, it might be disconcerting to the reader that I have left the derivatives on the LHS as total, rather than partial. Well, that's the way to cookie crumbles in SD. I'm just following SD rules. Since the functional dependence of ω was **not** stated explicitly, I must leave the derivative as total. However, I can make this 'fair' interpretation: Let's assume that ω is explicitly dependent on $\mathbf{x}$, and on $\mathbf{x}$ only. Then, the original presentation of $\omega = g(\mathbf{u}(\mathbf{x}))$, is more properly stated as

$$\omega(\mathbf{x}) = g(\mathbf{u}(\mathbf{x}))\,. \tag{8}$$

With this amendment to the problem statement, we can write

$$\left[\frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y}, \frac{\partial\omega}{\partial z}\right] = \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right] \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{bmatrix}\,. \tag{9}$$

So, we're ready to do what calculations we can and enter them in the matrix equation, yielding:

$$\left[\frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y}, \frac{\partial\omega}{\partial z}\right] = \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right] \begin{bmatrix} \dfrac{1}{y} & \dfrac{-x}{y^2} & 0 \\ 0 & \dfrac{-z}{y^2} & \dfrac{1}{y} \end{bmatrix}\,. \tag{10}$$

Next, we can do some extractions from this equation.

$$\frac{\partial \omega}{\partial x} = \frac{1}{y}\frac{\partial g}{\partial u}\,, \tag{11a}$$

$$\frac{\partial \omega}{\partial y} = \frac{-x}{y^2}\frac{\partial g}{\partial u} - \frac{z}{y^2}\frac{\partial g}{\partial v}\,, \tag{11b}$$

$$\frac{\partial \omega}{\partial z} = \frac{1}{y}\frac{\partial g}{\partial v}\,. \tag{11c}$$

Hence, the value of the expression we seek is

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial x} + z\frac{\partial \omega}{\partial z} = \frac{x}{y}\frac{\partial g}{\partial u} + y\left(\frac{-x}{y^2}\frac{\partial g}{\partial u} - \frac{z}{y^2}\frac{\partial g}{\partial v}\right) + z\frac{1}{y}\frac{\partial g}{\partial v} \tag{12a}$$

$$= \frac{x}{y}\frac{\partial g}{\partial u} - \frac{x}{y}\frac{\partial g}{\partial u} - \frac{z}{y}\frac{\partial g}{\partial v} + \frac{z}{y}\frac{\partial g}{\partial v} \tag{12b}$$

$$= 0\,. \tag{12c}$$