Math Diversion Problem 431

P. Reany

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Every big idea needs someone to defend it.
— Cybersecurity

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=wL-P09_6T1U
Title: An Interesting Equation
| Problem 482
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^i = i^z \,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{5}$$

3 The Solution

Before I begin, let's look at the solution as given by WolframAlpha: $z = -i$. I suppose that this could be called the 'trivial' solution, but my answer is somewhat different.

The natural action to begin with is to take the logarithm across (1):

$$i \ln z = z \ln (i \cdot e^{2\pi i n}) = z \ln (e^{i\pi/4 + 2\pi i n}) = z i(\pi/4 + 2\pi n) \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{6}$$

Next, multiply through by i and divide through by z:

$$-z^{-1} \ln z = -(\pi/4 + 2\pi n) \quad \text{for} \quad n \in \mathbb{Z}\,, \tag{7}$$

which can be rewritten as

$$z^{-1} \ln z^{-1} = -(\pi/4 + 2\pi n) \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{8}$$

Now we can take the Lambert W function across this equation.

$$\ln z^{-1} = W(-(\pi/4 + 2\pi n)) \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{9}$$

So, finally, we get for z:

$$z = e^{-W(-(\pi/4+2\pi n))} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{10}$$