Math Diversion Problem 432: Euler's theorem on homogeneous polynomial functions

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> Time and again I come across people who, although very bright, and having received advanced mathematical education, have a completely confused notion of what partial differentiation is.
> — andrewkirk

> I am having some issues understanding what should I keep constant and what not in certain cases when I take partial derivatives.
> — A poster to stackexchange

1 The Theorem

Prove Euler's theorem on homogeneous polynomial functions.

The theorem goes like this: Let $f(x_1, x_2, \dots, x_r)$ be a homogeneous polynomial function od degree n of r independent variants $\{x_1, x_2, \dots, x_r\}$.[1] Prove that the following relation holds true:

$$\sum_{i=1}^{r} x_i \frac{\partial f(x_1, x_2, \dots, x_r)}{\partial x_i} = n f(x_1, x_2, \dots, x_r) \,. \tag{1}$$

2 The Preparation

First, we except as fact that polynomials are differentiable. Second, let's investigate what a homogeneous polynomial really is by example. Consider the polynomial of second degree

$$f(x, y) = x^2 + 3xy + 5y^2 \,. \tag{2}$$

[1] A *variant* of a function is a variable on which the function is explicitly dependent.

This polynomial is said to be homogeneous of degree 2 because each term is of degree 2. So, what would happen if we replaced $x \to tx$ and $y \to ty$, then

$$f(tx, ty) = (tx)^2 + 3(tx)(ty) + 5(ty)^2 = t^n(x^2 + 3xy + 5y^2) = t^n f(x, y)\,. \tag{3}$$

3 The Proof

I'll use Structured Differentiation (SD) to prove this theorem. The variant of f will be represented by the vector $\mathbf{x}$. In SD, the theorem takes the form

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} = n f(\mathbf{x})\,, \tag{4}$$

where we are to interpret this as a matrix equation, where $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}$ is a row vector and $\mathbf{x}$ is a column vector.

Now, we have to replace each component x_i in $\mathbf{x}$ with tx_i; that is, $\mathbf{x} \to t\mathbf{x}$. Then we may write

$$f(t\mathbf{x}) = t^n f(\mathbf{x})\,. \tag{5}$$

Now, we just take the total derivative with respect to t across this equation, to get on the LHS

$$\frac{\delta f(t\mathbf{x})}{\delta t} = \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}}\frac{\delta \mathbf{u}}{\delta t}\Big|_{\mathbf{u}=t\mathbf{x}} = \frac{\partial f(t\mathbf{x})}{\partial (t\mathbf{x})}t\mathbf{x}\,. \tag{6}$$

And the RHS becomes

$$\frac{\delta(t^n f(\mathbf{x}))}{\delta t} = nt^{n-1} f(\mathbf{x})\,. \tag{7}$$

Now, all we have to do is to set the LHS equal to the RHS and let $t \to 1$, and we have that

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} = n f(\mathbf{x})\,. \tag{8}$$

Comment: So, what's magical about setting $t = 1$ to get this result? Nothing, really. The point is that (6) and (7) must be true for all real values of t, including for $t = 1$. It's just that for $t = 1$, we get the result we're after.