Math Diversion Problem 433

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There is much you have to learn. You must explore; you
must reach out. Go...and give thought to the
the mysteries of the universe.
— The Galaxy Being
(An early proponent of
Life-Long Learning)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=8GXCTh8vg4k
Title: An Absolutely Nice Homemade Equation | Problem 473
 Presenter:  aplusbi
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1 The Problem

Given the relation

$$\frac{1}{z} + \frac{1}{|z|} = z_0 = \frac{9+3i}{25}\,, \tag{1}$$

find the values of z.

Note: $z = a + bi$, $z\overline{z} = r^2 = |z|^2$, and $z_0 = a_0 + b_0 i$.

2 The Solution

First, let's clear of fractions by multiplying through by $r^2 = z\overline{z}$ with simplifications:

$$\frac{1}{z} z\overline{z} + \frac{1}{|z|} z\overline{z} = z_0 z\overline{z}\,, \tag{2}$$

or

$$\overline{z} + r = r^2 z_0\,. \tag{3}$$

Now, let's conjugate through.

$$z + r = r^2 \overline{z}_0\,. \tag{4}$$

Solving this for z, we have that

$$z = r^2 \overline{z}_0 - r\,. \tag{5}$$

So, if we knew the value of r, we would know z. To that end, let's multiply (5) with its conjugate, to get

$$r^2 = r^4 z_0 \overline{z}_0 - r^3 z_0 - r^3 \overline{z}_0 + r^2\,, \tag{6}$$

which simplifies to

$$r^3(r\,r_0^2 - 2a_0) = 0\,. \tag{7}$$

Now, since $r \neq 0$, then $r\,r_0^2 - 2a_0 = 0$, from which we have that

$$r = \frac{2a_0}{r_0^2} = \frac{2(9/25)}{90/25^2} = 5\,. \tag{8}$$

Therefore, from (5), we get

$$z = 25\,\frac{9-3i}{25} - 5 = 4 - 3i\,. \tag{9}$$