Math Diversion Problem 440: Partial Derivatives Using SD

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Abstract

Structured Differentiation (SD) is used to solve a problem I found on a physics forum.

To err is human; to really foul things up
requires partial differentiation.
— my adaptation of a quote from
Paul Ehrlich

1 Introduction

This problem is from August 5, 2017.

> `https://www.physicsforums.com/threads/partial-derivative-homework-calculate-f-x.921970/`

The problem statement:
The question asks to calculate $\partial f/\partial x$ for

$$f(x, y, t) = 3x^2 + 2xy + y^{1/2}t - 5xt \tag{1}$$

where

$$x(t) = t^3 \quad \text{and} \quad y(t) = 2t^5\,. \tag{2}$$

The poster provides the answer given as

$$\frac{\partial f}{\partial x} = 6x + 2y - 5t\,. \tag{3}$$

Then the poster explained:

> I'm confused because the answer given seems to treat x, y, t as independent variables and the answer given is just a partial derivative treating y and t as constant. But really x, y, t are all dependent on each other. Is it even possible to obtain a partial derivative with respect to x in this case?

My reply to this question is, Yes it's very possible to take the 'partial derivative with respect to x in this case'. It's quite obvious to me that the derivative taken of (1) to obtain (3) is the 'explicit derivative' of f by x. Physics books are more likely to teach about the 'explicit derivative' than are math books, yet, when I see these so-called 'partial derivatives' defined in math books, they typically explain the process as: Let the function vary with respect to that one variable, treating all other variables as constants. And that goes directly to the point of the poster's question. I suppose that part of the confusion people have concerning the symbol $\partial f/\partial x$ is when to interpret it as an explicit derivative and when to interpret it as a total derivative.

So, what good are explicit derivatives? Besides their use in the components of gradients, they are useful for computing total derivatives in a step-wise fashion. I'll demonstrate this now by taking the total derivative of $f(x, y, t)$ by t.

First, we set $f(x, y, t) = f(\mathbf{x})$, where $\mathbf{x} = (x, y, t)^{\mathrm{T}}$. Then, the total derivative of f by t is given by use of the *chain rule*:

$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial t}\right]\begin{bmatrix}\dfrac{dx}{dt}\\ \dfrac{dy}{dt}\\ \dfrac{dt}{dt}\end{bmatrix} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\,, \tag{4}$$

where $dt/dt = 1$. Now, the first two terms on the RHS of (4) tell us about how f changes by t implicitly through x and y, respectively. In SD there is a simple symbol to represent them collectively, namely, $\dfrac{\not{\partial} f}{\not{\partial} t}$, which is called the 'copartial derivative by t'. The remaining term on the RHS tells us how f varies through t explicitly, and given the name the 'partial derivative by t'. Thus, in SD the partial derivative is always explicit.

So, by using the shorthand of SD, we can rewrite (4) into the generic form

$$\frac{df}{dt} = \frac{\not{\partial} f}{\not{\partial} t} + \frac{\partial f}{\partial t}\,. \tag{5}$$

Now, let's use a trick that mathematicians love to use, but physicists not so much. Introduce the function g, such that

$$f(x, y, t) = g(t)\,, \tag{6}$$

where g is the result of replacing x and y by their expressions in terms of t. Thus, g is only a function of t, and on performing the total derivative by t across (6), we get that

$$\frac{df}{dt} = \frac{dg}{dt} = \frac{\partial g}{\partial t}\,, \tag{7}$$

where the implicit derivative of g by t is identically zero, of course. So, in this special case, the partial derivative of g by t is both explicit and total, but only because the implicit derivative is identically zero.

Definition: The **variants** of a function are the variables on which the function is explicitly dependent. In the case of f defined in (1), its variants are x, y, t.

Definition: A function is said to be in **primitive form** or just **primitive** if none of its variants are functionally dependent on any other variant.

Thus, in (6), we say that g is primitive in t and f has been reduced to primitive form.

2 Using the partial derivative within the chain rule

So, I want to finish this article by performing the total derivative of f by t in the step-wise fashion given in (4):

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial t}\right] = \left[6x + 2y - 5t,\ 2x + \tfrac{1}{2}y^{-1/2}t,\ y^{1/2} - 5x\right], \tag{8}$$

and

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dt}{dt} \end{bmatrix} = \begin{bmatrix} 3t^2 \\ 10t^4 \\ 1 \end{bmatrix}. \tag{9}$$

Plugging this stuff into (4), we get

$$\begin{aligned} \frac{df}{dt} &= \left[6x + 2y - 5t,\ 2x + \tfrac{1}{2}y^{-1/2}t,\ y^{1/2} - 5x\right] \begin{bmatrix} 3t^2 \\ 10t^4 \\ 1 \end{bmatrix} \\ &= (6x + 2y - 5t)(3t^2) + (2x + \tfrac{1}{2}y^{-1/2}t)(10t^4) + (y^{1/2} - 5x). \end{aligned} \tag{10}$$

I'll leave it at that, but if you want, you can replace x and y with their values in terms of t.

3 Conclusion

I invented SD as a way to remove the confusion out of 'partial differentiation'. One way to do that is to embrace matrices as a means of bringing order out of the chaos. The future should bring more such examples of its use presented on this forum.