Math Diversion Problem 443

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You cannot ask us to take sides against arithmetic.
— Winston Churchill

The problem is found at:

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Source: Abstract Algebra: A First Course (1992)
 Section 2: Problem 2.1 h.
 Presenter:  Dan Saracino
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1 The Problem

Given the set $S \equiv \mathbb{R} - \{1\}$ with binary operation defined by

$$a * b = a + b - ab\,, \tag{1}$$

determine if this set forms a group.

2 The Solution

For $(S, *)$ to form a group, it needs to satisfy the following conditions:
1) The set needs a closed binary operation.
2) The set needs an identity element.
3) The set needs an inverse for every element.
4) The binary operation $*$ needs to be associative.

For $*$ to be closed, it needs to take in a pair of elements of S and map it to another member of S. That means that the image element can be any real number except 1. So, can it ever be unity? What values of a and/or b would yield unity? Let's see:

$$a + b - ab = 1\,, \tag{2}$$

can be rewritten as

$$a(1 - b) = 1 - b\,, \tag{3}$$

which would require either $a = 1$ or $b = 1$, which are both disallowed, so that $*$ is a closed binary operator.

Next, what is the identity element of S? It's 0 and I'll prove it.

$$a * 0 = a + 0 - a0 = a\,, \tag{4a}$$

and

$$0 * b = 0 + b - 0b = b\,. \tag{4b}$$

Next, what is an element's inverse?

$$a * b = a + b - ab = 0\,. \tag{5}$$

Solving for b, we get

$$b = \frac{a}{a-1}\,. \tag{6}$$

We can test this:

$$a * \frac{a}{a-1} = a + \frac{a}{a-1} - \frac{a^2}{a-1} = 0\,. \tag{7}$$

To establish b^{-1} is similar.

Lastly, is $*$ an associative operator? Specifically, what we must show is that

$$(a * b) * c = a * (b * c)\,. \tag{8}$$

Starting with the LHS:

$$\begin{aligned}(a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - (a + b - ab)c \\ &= a + b + c - ab - ac - bc + abc\,. \end{aligned} \tag{9}$$

Next with the RHS:

$$\begin{aligned}a * (b * c) &= a + (b * c) - a(b * c) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc\,. \end{aligned} \tag{10}$$

Therefore, $*$ is an associative operator. Thus, $(S, *)$ is a group.