

Math Diversion Problem 444: Partial Derivatives Using SD

P. Reany

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Abstract

Structured Differentiation (SD) is used to solve a problem I found on a physics forum.

To err is human; to really foul things up
requires partial differentiation.
— my adaptation of a quote from
Paul Ehrlich

1 Introduction

This problem is from May 5, 2022.

<https://www.physicsforums.com/threads/calculating-the-partial-derivative-in-polar-coordinates.1014922>

The problem statement:

The question asks to calculate the conversion of the Laplacian in 2 dimensions from rectangular coordinates x, y to polar coordinates, r, θ , where

$$x = r \cos \theta, \quad (1a)$$

$$y = r \sin \theta, \quad (1b)$$

and where

$$z = f(x, y), \quad (2)$$

to show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2. \quad (3)$$

Solution.

I'm going to start off by bending the notation rules a bit, but it should work. Let

$$z(\mathbf{u}) = z(\mathbf{x}(\mathbf{u})), \quad (4)$$

where

$$\mathbf{x} = (x, y)^T \quad \text{and} \quad \mathbf{u} = (r, \theta)^T, \quad (5)$$

where 'T' is for transpose, then

$$\frac{\partial z(\mathbf{u})}{\partial \mathbf{u}} = \frac{\partial z}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}}. \quad (6)$$

To explain what I did, I applied the total derivative by \mathbf{u} across (4) and then dropped the implicit parts of the derivatives to end up with (6). Now, the only real work to be done is in a) evaluating the partials in $\partial \mathbf{x} / \partial \mathbf{u}$ and then b) performing a matrix inverse. So, since x, y are easy functions of r, θ , as given by (1a) and (1b), we can expand (6) rather easily.

$$\left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta} \right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}. \quad (7)$$

Now we need to solve for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ so that we can substitute those values into the LHS of (3) to produce its RHS. To accomplish this, we solve for them as the components of a 1×2 matrix:

$$\left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta} \right) \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}^{-1} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right), \quad (8)$$

or, even better:

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = \left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta} \right) \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \frac{1}{r}, \quad (9)$$

where

$$r = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}. \quad (10)$$

Solving for the partials we seek, we have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial z}{\partial \theta} \sin \theta, \quad (11a)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \sin \theta + \frac{1}{r} \frac{\partial z}{\partial \theta} \cos \theta. \quad (11b)$$

Now, referring back to (3):

$$\begin{aligned} \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 &= \left[\frac{\partial z}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial z}{\partial \theta} \sin \theta \right]^2 + \left[\frac{\partial z}{\partial r} \sin \theta + \frac{1}{r} \frac{\partial z}{\partial \theta} \cos \theta \right]^2 \\ &= \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2. \end{aligned} \quad (12)$$