Math Diversion Problem 446: Partial Derivatives Using SD

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> Thermodynamics is the study of the laws which govern
> transformations of matter and energy during
> physical and chemical changes.
> — S. M. Blinder[1]

1 Introduction

Long ago I took some notes from a paper by Blinder, to which now it seems appropriate for me to present some of that content here, as it is relevant to partial differentiation.

Let's begin with a simple functional dependency: $z = z(x.y)$. Then,

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy\,, \tag{1}$$

where the subscript is telling us which variable is being held constant during the differentiation.

Now, what if we add the constraint

$$z = z(x, y) = \text{const}\,? \tag{2}$$

Then, all the derivatives of z should be identically zero, right? Well, no. Certainly the total derivative of z by x or y would be zero, but not necessarily the 'partial derivatives' by its variants.[2]

With the added constraint (2), (1) becomes

$$\left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy = 0\,, \tag{3}$$

[1] Found in "Mathematical Methods in Elementary Thermodynamics," S. M. Blinder, J. of Chem. Ed., Vol. 43, No. 2, 1966, pp 85–88.

[2] A variant of a function is a variable on which the function is explicitly dependent.

On solving this for dy/dx, we get

$$\left.\frac{dy}{dx}\right|_{z=\text{const}} = -\frac{\left(\dfrac{\partial z}{\partial x}\right)_y}{\left(\dfrac{\partial z}{\partial y}\right)_x} . \tag{4}$$

We can now make two subtle but important changes to this last equation. First, the total derivative on the LHS can be replaced by a partial derivative, namely,

$$\left.\frac{dy}{dx}\right|_{z=\text{const}} = \left(\frac{\partial y}{\partial x}\right)_z . \tag{5}$$

And, second, we can go the other way. From the denominator in (4),

$$\left(\frac{\partial z}{\partial y}\right)_x = \left.\frac{dz}{dy}\right|_{x=\text{const}} . \tag{6}$$

And if you remember your ordinary differentiation, then

$$\frac{1}{\left(\dfrac{\partial z}{\partial y}\right)_x} = \left(\left.\frac{dz}{dy}\right|_{x=\text{const}}\right)^{-1} = \left.\frac{dy}{dz}\right|_{x=\text{const}} = \left(\frac{\partial y}{\partial z}\right)_x . \tag{7}$$

On putting all this together, (4) can be rewritten as

$$\left(\frac{\partial y}{\partial x}\right)_z = -\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x . \tag{8}$$

And with just a bit more slight of hand, we can declare the enigmatic relation:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x = -1 . \tag{9}$$

I did this the long way, but now the short way, because we need to understand how they go together. Combining (4) and (5), we get

$$\left(\frac{\partial y}{\partial x}\right)_z = -\left(\frac{\partial z}{\partial x}\right)_y \Big/ \left(\frac{\partial z}{\partial y}\right)_x . \tag{10}$$

yielding

$$\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial x}{\partial z}\right)_y = -1 , \tag{11}$$

which is Blinder's result, but it is equivalent to (9) under the swapping of x and y. This makes sense because all the way back in (3), x and y entered the problem symmetrically.