Math Diversion Problem 447

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No mystery is closed to an open mind.
— Tim White
TV show *Sightings*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=-Cg0niL4whY
Title: Can you Pass Stanford University Admission Test ?
 Presenter:  Super Academy
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1 The Problem

Given the relation

$$5^{\sqrt{x}+1} + 5\sqrt{x} = 135\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function.

3 The Solution

The first thing I'll do is to perform a change of variable. Let

$$y = \sqrt{x}\,. \tag{9}$$

Then (1) becomes

$$5^{y+1} + 5y = 135\,, \tag{10}$$

which can be transformed algebraically to

$$5^{y+1} = 135 - 5y\,. \tag{11}$$

Next, we'll let

$$z = 135 - 5y\,. \tag{12}$$

Then (11) becomes

$$5^{28-z/5} = z\,, \tag{13}$$

and then

$$5^{28} = z5^{z/5}\,, \tag{14}$$

In a more canonical form, this becomes

$$(z/5)\,5^{z/5} = 5^{27}\,. \tag{15}$$

Next, we apply the Lambert W function across this equation.

$$z/5 = W_5(5^{27}) = \frac{W_n(5^{27} \cdot \ln 5)}{\ln 5}\,, \tag{16}$$

[1]This notation I invented myself.

or

$$z = 5\frac{W_n(5^{27} \cdot \ln 5)}{\ln 5}\,. \tag{17}$$

For the one real value of z, which I'll refer to as z_0, we have

$$z_0 = 5\frac{W_0(5^{27} \cdot \ln 5)}{\ln 5}\,. \tag{18}$$

In the lemmas above, it was shown that

$$W_0(a \ln a) = \ln a\,. \tag{19}$$

Perhaps we can use this simplification to get a simpler form for z_0. To accomplish that, we'll need to factor $5^{27} = 5^{27-m}5^m$

m	5^{27-m}
5	5^{22}
10	5^{17}
20	5^{7}
25	5^{2} ✓

Table 1: Heuristic: Solved by Table.

Hence, $a = 2^{25}$. Then,

$$z_0 = 5\frac{\ln 5^{25}}{\ln 5} = 5\frac{25 \ln 5}{\ln 5} = 125\,. \tag{20}$$

On solving (12) for y, we get

$$y = 27 - z_0/5 = 2\,. \tag{21}$$

So, finally,

$$x = y^2 = 4\,. \tag{22}$$