Math Diversion Problem 450

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> Unless you try to do something beyond what you have
> already mastered, you will never grow.
> — Ralph Waldo Emerson

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=UjlyqwBJZic
Title: An Interesting Transcendental Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{\sqrt{3}} = \sqrt{3}^x \,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function.

3 The Solution

$$x^{\sqrt{3}} = \sqrt{3}^x\,, \tag{9}$$

The first thing I'll do is to isolate x on the LHS to its first power by taking the appropriate roots across (9):

$$x = \left(\sqrt{3}^x\right)^{1/\sqrt{3}} = \left(\sqrt{3}^{1/\sqrt{3}}\right)^x\,, \tag{10}$$

or

$$x = a^x\,, \tag{11}$$

where

$$a \equiv \sqrt{3}^{1/\sqrt{3}}\,. \tag{12}$$

Now, (11) can be rewritten as

$$xa^{-x} = 1\,, \tag{13}$$

which can be put into canonical form as[2]

$$-xa^{-x} = -1\,. \tag{14}$$

Then, employing one of the lemmas:

$$-x = W_a(-1) = \frac{W_n(-1 \cdot \ln a)}{\ln a}\,. \tag{15}$$

[1]This notation I invented myself.

[2]By 'canonical form', I mean a form just right for applying one of the Lambert lemmas.

Hence,

$$x = -\frac{W_n(-\ln \sqrt{3}^{1/\sqrt{3}})}{\ln \sqrt{3}^{1/\sqrt{3}}} = -\frac{W_n(\frac{1}{\sqrt{3}} \ln \frac{1}{\sqrt{3}})}{(1/\sqrt{3}) \ln \sqrt{3}} \,. \tag{16}$$

At this point, I'd like to reveal what WolframAlpha declares are the two real solutions:

$$x = 1.73205 \tag{17}$$

$$x = 5.19615 \tag{18}$$

In fact, these are all the solutions that WolframAlpha admits to.

Now, our possible real solutions will come from $n = 0$ and $n = -1$.

$$x_0 = -\frac{W_0(\frac{1}{\sqrt{3}} \ln \frac{1}{\sqrt{3}})}{(1/\sqrt{3}) \ln \sqrt{3}} \,, \tag{19}$$

$$x_{-1} = -\frac{W_{-1}(\frac{1}{\sqrt{3}} \ln \frac{1}{\sqrt{3}})}{(1/\sqrt{3}) \ln \sqrt{3}} \,. \tag{20}$$

I'm not going to calculate x_{-1}, but I will calculate x_0. Applying (5), we get

$$x_0 = -\frac{\ln \frac{1}{\sqrt{3}}}{(1/\sqrt{3}) \ln \sqrt{3}} = \sqrt{3} \approx 1.73205 \,. \tag{21}$$

By the way, $\frac{1}{\sqrt{3}} \ln \frac{1}{\sqrt{3}} \approx -0.31714$, which is larger than $-1/e \approx -0.36788$ — the smallest value the argument of the principal value of W can take.