

Math Diversion Problem 451

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When I give this talk to a physics audience, I
remove the quotes from my ‘Theorem’.
— Brian Greene

The YouTube video is found at:

Source: ??
Title: ??
Presenter: SyberMath

1 The Problem

Given the relation

$$e^x - 1 = \ln(x + 1), \quad (1)$$

find the values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \quad (2)$$

then

$$z = W(B), \quad (3)$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following:
If

$$y \ln y = B, \quad (4)$$

then

$$\ln y = W(y \ln y) = W(B). \quad (5)$$

The following is the ‘Lambert W function base s ’,¹ or W_s , where s is a positive real number. Let’s begin with the relation

$$xs^x = A, \quad (6)$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}. \quad (7)$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A), \quad (8)$$

which is the usual Lambert W function.

3 The Solution

How should I approach this problem? My method of solution generally relies on employing general techniques, rather than by looking for a trick that works in a special situation. On the other hand, it may be well for those who take competitive test to look for tricks, which might save a lot of time.

This is how I tend to approach this kind of problem: Transform the Given relation into a canonical form, which is a form ready to apply a lemma that will simplify the situation. If that’s not clear, this problem is an example of that heuristic.

Let’s begin with finding a trivial solution, if we can. How about $x = 0$? Yes, that works, but is there some other?

Let’s make the variable transformation:

$$e^{w-1} = x + 1, \quad (9)$$

and substitute that into (1), to get

$$e^{e^{w-1}-1} - 1 = w - 1, \quad (10)$$

which simplifies down to

$$e^{e^{w-1}-1} = w. \quad (11)$$

This can be transformed to

$$e^{-1}e^{e^{-1}e^w} = w. \quad (12)$$

¹This notation I invented myself.

Next, we'll multiply through by e^w , to get

$$e^{-1}e^w e^{e^{-1}e^w} = we^w, \quad (13)$$

and then take the Lambert W function across this, to get

$$e^{-1}e^w = w, \quad (14)$$

which can be algebraically manipulated to

$$-we^{-w} = -\frac{1}{e}. \quad (15)$$

On taking the principal W function across this, we get

$$-w = -1. \quad (16)$$

Using $w = 1$ in (9), we have that

$$x = 0. \quad (17)$$

So, was all this extra effort worth it? Maybe. If we had not caught the trivial solution at the beginning, we would have found it at the end.