Math Diversion 452: An Instructive Integral

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> The shortest path between two truths in the real domain
> passes through the complex domain.
> — Jacques Hadamard

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ynIDyOWGwBY
Title: An Interesting Integral
 Presenter: SyberMath
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1 The Problem

Find the indefinite integral

$$I = \int e^x \cos x \, dx \,. \tag{1}$$

Note: $\dfrac{1}{1+i} = \dfrac{1-1}{2}$.

2 The Solution

The trick to performing this integration elegantly is to step into the complex space and then step out of it with the answer.

The Given integral can be rewritten as

$$I = \mathrm{Re}\left[\int e^x (\cos x + i \sin x) \, dx \right], \tag{2}$$

where a pure imaginary term was added into the integrand, but ignored because of the operator in the front, named, 'Re', ignores it. Anyway, we continue.

$$
\begin{aligned}
I &= \mathrm{Re}\,\Big[\int e^x e^{ix}\,dx\Big] && \text{(3a)}\\
&= \mathrm{Re}\,\Big[\int e^{(1+i)x}\,dx\Big] && \text{(3b)}\\
&= \mathrm{Re}\,\Big[\frac{1}{1+i}\int e^{(1+i)x}\,d(1+i)x\Big] && \text{(3c)}\\
&= \mathrm{Re}\,\Big[\frac{1}{1+i}e^{(1+i)x}+C\Big] && \text{(3d)}\\
&= \mathrm{Re}\,\Big[\frac{1-i}{2}e^x e^{ix}+C\Big] && \text{(3e)}\\
&= \mathrm{Re}\,\Big[\frac{1-i}{2}e^x(\cos x+i\sin x)+C\Big] && \text{(3f)}\\
&= \frac{1}{2}e^x\mathrm{Re}\,\big[(1-i)(\cos x+i\sin x)+C\big] && \text{(3g)}\\
&= \frac{1}{2}e^x\,[\,\cos x+\sin x\,]+c\,, && \text{(3h)}
\end{aligned}
$$

where c is an arbitrary real number. Hence,

$$\int e^x\cos x\,dx = \frac{1}{2}e^x\,[\,\cos x+\sin x\,]+c\,. \tag{4}$$

So why did this technique work? It worked because in each step, the real and imaginary parts of the overall expression were preserved.