Math Diversion Problem 453

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Zathras is used to being beast of burden to other
people's needs. Very sad life... Probably have
very sad death. But, at least
there is symmetry.
—Zathras
(A character from *Babylon 5*)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=X5iYlxCwGRo&list
     =PLMvuVeOn1Hd_KIT-dsvIVluQQN3pJrlmX&index=46
Title: A Nice Exponential Equation
 Presenter: Master T Maths Class
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1 The Problem

Given the relation

$$x^{x^{1+x}} = 256\,, \tag{1}$$

find the values of x over the real numbers.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Today's important identity is:
$W(x^{x+1}\ln x) = x\ln x$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{5}$$

3 The Solution

I've already worked this problem before (# 148), using the α substitution. This time I want to use the Lambert identity on page one.

Note: $256 = 2^8 = 4^4$.

So, we'll begin by taking the logarithm across (1):

$$x^{1+x} \ln x = \ln 256 = 4 \ln 4 \,. \tag{6}$$

Next, we'll take the Lambert W function across this and apply the aforementioned identity, to get

$$x \ln x = W(4 \ln 4) = \ln 4 = 2 \ln 2 \,. \tag{7}$$

Now, applying the W function again, but this time using one of the lemmas, we have that

$$\ln x = \ln 2 \,. \tag{8}$$

Therefore,

$$x = 2 \,. \tag{9}$$