Math Diversion 454: Mathematical Induction

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> A more modern attempt to explain the fruitfulness of
> mathematical reasoning is that of Poincaré, who finds
> it all due to the principle of mathematical induction.
> —Morris R. Cohen
> ("The Present Situation in the
> Philosophy of Mathematics")

`Source: An induction problem in Fibonacci numbers`

1 The Problem

The Fibonacci numbers are defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \,, \tag{1}$$

where

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \,. \tag{2}$$

The Fibonacci numbers have the proposed closed-form solution of

$$F_n = \frac{a^n - b^n}{a - b} \quad \text{where} \quad n \geq 2 \,, \tag{3}$$

and where a and b ($b < a$) are distinct solutions to the quadratic

$$x^2 = x + 1 \,, \tag{4}$$

which gives us the two relations:

$$a^2 = a + 1 \,, \tag{5a}$$

$$b^2 = b + 1 \,. \tag{5b}$$

Our job now is to show that (3) is correct by use of mathematical induction.

2 The Solution

When doing a proof by induction, we begin by establishing that the formula holds for its 'base' case, that is, for the case of lowest n. By Eq. (3), we see that the base case occurs for $n = 2$. Okay, so what should we expect the formula to give us? For that, we go back to (1) and use $n = 0$:

$$F_2 = F_1 + F_0 = 1 + 0 = 1\,. \tag{6}$$

So, what does the formula say? Using $n = 2$ in Eq. (3), we see that

$$\begin{aligned} F_2 &= \frac{a^2 - b^2}{a - b} \\ &= \frac{(a+1) - (b+1)}{a - b} \\ &= \frac{a - b}{a - b} \\ &= 1\,. \end{aligned} \tag{7}$$

Okay, so far so good. Our next step is to invoke the so-called 'inductive hypothesis', which mean that we will assume the truth of (3) for the arbitrary index value n, and then use that together with the rest of the given information (where needed) to show that the formula holds as well when the index $n \to n+1$. In that case, we should be able to show that

$$F_{n+1} = \frac{a^{n+1} - b^{n+1}}{a - b}\,. \tag{8}$$

So, let's try for it!

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \frac{a^n - b^n}{a - b} + \frac{a^{n-1} - b^{n-1}}{a - b} \\ &= \frac{(a^n + a^{n-1}) - (b^n + b^{n-1})}{a - b} \\ &= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{a - b} \\ &= \frac{a^{n-1}a^2 - b^{n-1}b^2}{a - b} \quad \text{(from (5a) and (5b))} \\ &= \frac{a^{n+1} - b^{n+1}}{a - b}\,, \end{aligned} \tag{9}$$

which is what we were to show.