Math Diversion Problem 461

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If there is to be a brave new world, our generation is
going to have the hardest time living in it.
— Chancellor Gorkon (Star Trek VI,
The Undiscovered Country)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=xuugqqgG6d4
Title: Oxford entrance exam question
 Presenter: Math Beast
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1 The Problem

Given the relation

$$x^{1/x} = e^{\pi/2}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

3 The Solution

So, we'll begin by taking the logarithm across (1):

$$x^{-1} \ln x = \pi/2 + 2\pi i n \quad \text{for } n \in \mathbb{Z}\,. \tag{6}$$

Next, we multiply through by -1:

$$x^{-1} \ln x^{-1} = -\pi/2 - 2\pi i n \quad \text{for } n \in \mathbb{Z}\,. \tag{7}$$

Now we take the Lambert W function across this, to get

$$\ln x^{-1} = W(-\pi/2 - 2\pi i m) \quad \text{for } m \in \mathbb{Z}\,. \tag{8}$$

On raising e to the power of this last equation, we have that

$$x^{-1} = e^{W_n(-\pi/2 - 2\pi i m)} \quad \text{for } m \in \mathbb{Z}\,. \tag{9}$$

Finally,

$$x = e^{-W_n(-\pi/2 - 2\pi i m)} \quad \text{for } m \in \mathbb{Z}\,. \tag{10}$$

This answer is 'close' to WolframAlpha's answer, though it only gives values for W_{-1}, W_0, and W_1. It doesn't find any real solutions. And, as I am no expert on Lambert, I will leave this at this point.