Math Diversion 463

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Even the youths shall faint and be weary, and the young
men shall utterly fall: But they that wait upon the Lord
shall renew their strength; they shall mount up with wings
as eagles; they shall run, and not be weary; and
they shall walk, and not faint.
— Isaiah 40:30–31.

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Title: A Lambert $W$ Integral
 Presenter: Patrick
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1 The Problem

Find the indefinite integral

$$I = \int W(x)\, dx\,, \tag{1}$$

where W is the Lambert W function.

2 The Solution

$$I = \int W(x)\, dx\,. \tag{2}$$

I'm going to accept as given that

$$\frac{dW(x)}{dx} = \frac{W(x)}{x(1+W(x))}\,. \tag{3}$$

Now, let's take the derivative of $W(x)x$:

$$D_x(W(x)x) = W'(x)x + W(x)\,. \tag{4}$$

Next, we integrate this:

$$W(x)x = \int W'(x)x\,dx + \int W(x)\,dx = \int W'(x)x\,dx + I\,. \tag{5}$$

Now, we set $J = \int W'(x)x\,dx$, and when we have integrated this, then

$$I = W(x)x - J\,. \tag{6}$$

So,

$$\begin{aligned}
J &= \int W'(x)x\,dx && (7a)\\
&= \int \frac{W(x)}{x(1+W(x))}\,x\,dx && (7b)\\
&= \int \frac{W(x)}{1+W(x)}\,\frac{dx}{dW}\,dW && (7c)\\
&= \int \frac{W(x)}{1+W(x)}\left(\frac{dW}{dx}\right)^{-1} dW && (7d)\\
&= \int \frac{W(x)}{1+W(x)}\,\frac{x(1+W(x))}{W(x)}dW && (7e)\\
&= \int x\,dW && (7f)\\
&= \int We^{W}\,dW && (7g)\\
&= (W-1)e^{W} + C && (7h)\\
&= (1-\frac{1}{W})We^{W} + C && (7i)\\
&= x(1-\frac{1}{W}) + C\,, && (7j)
\end{aligned}$$

where C is an arbitrary complex number. Hence,

$$\begin{aligned}
I &= W(x)x - x(1-\frac{1}{W}) + C && (8a)\\
\int W(x)\,dx &= x\big[\,W(x) - 1 + \frac{1}{W(x)}\,\big] + C\,. && (8b)
\end{aligned}$$

Comment: A proof this long probably has a shorter version.