Math Diversion 464

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Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=pLbG3Vvpjjk&list
  =PLql0qQWQbo6ko_ESCoqvW4dkLMb8eHN9S
Title: Prove that 11^n - 4^n is divisible by 7 for any
    natural number
 Presenter: Prime Newtons
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1 The Problem

Use induction to show that 7 divides

$$11^n - 4^n\,, \tag{1}$$

for every n a positive integer.

2 The Preparation

Let a and b be positive integers. Then the expression '$a \,|\, b$' is the assertion that a divides b, which means that there exists some positive integer β such that

$$b = a\beta\,. \tag{2}$$

3 The Solution

For an induction proof, we begin by establishing the base case, which for this problem is for $n = 1$:

$$11^1 - 4^1 = 7\,, \tag{3}$$

which is certainly divisible by 7.

Next, we assume that the given hypothesis is true for some arbitrary positive integer k, yielding the assertion:

$$7 \mid (11^k - 4^k)\,, \tag{4}$$

and then we are to use this to show that

$$7 \mid (11^{k+1} - 4^{k+1})\,. \tag{5}$$

So, let's get started.

$$\begin{aligned} 11^{k+1} - 4^{k+1} &= 11 \cdot 11^k - 4^{k+1} && \text{(6a)} \\ &= (7+4) \cdot 11^k - 4 \cdot 4^k && \text{(6b)} \\ &= 7 \cdot 11^k + 4(11^k - 4^k)\,. && \text{(6c)} \end{aligned}$$

Now, by the inductive hypothesis, we've assumed that $7 \mid (11^k - 4^k)$; therefore

$$7 \mid (7 \cdot 11^k + 4(11^k - 4^k))\,, \tag{7}$$

since 7 divides both terms separately. Therefore, we've established (5), and we are done.