Math Diversion Problem 465

P. Reany

March 15, 2025

People often overlook the obvious.
— Doctor Who

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=NfLc-l5ghr0
Title: South Korea - A Nice Radical Equation
 Presenter: Maths & Olympiad
```

1 The Problem

Given the relation

$$\sqrt[8]{x} + \sqrt[12]{x} = 150\,, \tag{1}$$

find the real values of x.

2 The Preparation

The 'least common multiple' (LCM) of two positive integers a, b is the smallest positive integer that both a and b divide evenly. The notation for this is $\text{LCM}(a, b)$.

Now, for any choices for a and b positive integers, a common multiple of them both is, obviously, their product, ab. But there are times that what we really want is the **smallest** number that they both divide evenly, or, the LCM of a and b.

Let's do an example. Consider $a = 8$ and $b = 12$, which are suggested by the problem we're trying to solve at the moment. Their product is $ab = 8 \times 12 = 96$. Obviously, both 8 and 12 will divide 96 evenly. But can we do 'better'? Yes, the $\text{LCM}(8, 12) = 24$. The problem at hand is an example of why we would prefer to use the $\text{LCM}(8, 12)$, rather than just the common product.

3 The Solution

My inclination is to perform a change of variable to convert the given 'root' equation into a polynomial equation. So, let's set

$$x = y^{24}\,, \tag{2}$$

where, as we've just seen, $24 = \mathrm{LCM}(8, 12)$. Then, the Given equation converts to

$$y^3 + y^2 = 150\,. \tag{3}$$

But if we had settled for the product of 8 and 12, which is 96, our converted equation would have been

$$y^{12} + y^8 = 150\,. \tag{4}$$

I don't know about the reader, but I prefer to work with (3).

Now, I'm inclined to believe that (3) has at least one integer solution, so let's go find it (if it exists).

y	$y^3 + y^2$
4	$4^3 + 4^2 = 64$
5	$5^3 + 5^2 = 150$ ✓

Table 1: An integer root does exist!

So, we now have one root for x, namely, $x = 5^{24}$. But if we use long division on $y^3 + y^2 - 150$ by $y - 5$, we get

$$\frac{y^3 + y^2 - 150}{y - 5} = y^2 + 6y + 30\,, \tag{5}$$

which has no real roots. Hence, our final answer is

$$x = 5^{24}\,. \tag{6}$$