

Math Diversion 467

P. Reany

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Physical concepts are free creations of the human mind,
and are not, however it may seem, uniquely
determined by the external world.

— Albert Einstein

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=H6k4jvzIxn4>

Title: Prove $3^n + 7^n - 2$ is divisible by 4

Presenter: Prime Newtons

1 The Problem

Use induction to show that 4 divides

$$3^n + 7^n - 2, \tag{1}$$

for every n a positive integer.

2 The Preparation

Let a and b be positive integers. Then the expression ' $a | b$ ' is the assertion that a divides b , which means that there exists some positive integer β such that

$$b = a\beta. \tag{2}$$

3 The Solution

For an induction proof, we begin by establishing the base case, which for this problem is for $n = 1$. True or false:

$$4 | (3^1 + 7^1 - 2)? \tag{3}$$

Ans: True.

Next, we assume that the given hypothesis is true for some arbitrary positive integer k , yielding the assertion:

$$4 \mid (3^k + 7^k - 2), \quad (4)$$

and then we are to use this to show that

$$4 \mid (3^{k+1} + 7^{k+1} - 2). \quad (5)$$

So, let's get started.

$$3^{k+1} + 7^{k+1} - 2 = 3 \cdot 3^k + 7 \cdot 7^k - 2 \quad (6a)$$

$$= 3 \cdot 3^k + (3 + 4) \cdot 7^k - 2 \quad (6b)$$

$$= 3(3^k + 7^k) + 4 \cdot 7^k - 2 \quad (6c)$$

$$\stackrel{\text{VE}}{=} 3(3^k + 7^k - 2) + 6 + 4 \cdot 7^k - 2 \quad (6d)$$

$$= 3(3^k + 7^k - 2) + 4 \cdot 7^k + 4. \quad (6e)$$

Now, by the inductive hypothesis shown in (4), we've assumed that 4 divides the first term on the RHS. And, since it also divides the other two terms, then it will divide the entire RHS, which is all we needed to show.

As for that odd symbol over the equal sign in line (6d), it refers to **virtual emplacement**, which is a technique of getting an expression into the desired form by adding or multiplying it by suitable quantity.¹ I had to get the expression $3^k + 7^k - 2$ to be able to use (4), so I inserted the -2 inside $(3^k + 7^k)$. However, this has the side effect of subtracting $3 \times 2 = 6$ from the RHS; so, to compensate for this, I added 6 into the RHS as well.

¹These are the two main ways of performing a virtual emplacement, but there are other ways as well.