Math Diversion 468

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> The Axiom of Choice is obviously true; the Well Ordering
> Principle is obviously false; and who can tell
> about Zorn's Lemma?
> — Jerry Bona

Current problem: A typical problem in trigonometry.

1 The Problem

1) Use complex numbers to show that

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi\,, \tag{1a}$$

$$\sin(\theta + \varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi\,. \tag{1b}$$

2) And then use these results to show that

$$\tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\,\tan\varphi}\,, \tag{2}$$

2 The Preparation

$$e^{ix} = \cos x + i\sin x\,, \tag{3}$$

and

$$e^{i(x+y)} = e^{ix}e^{iy}\,. \tag{4}$$

3 The Solution

1) Using (3), with $x = \theta + \varphi$, we have that

$$e^{i(\theta+\varphi)} = \cos(\theta + \varphi) + i\sin(\theta + \varphi)\,, \tag{5}$$

and using (4), we have that

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi} \tag{6a}$$

$$= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \tag{6b}$$

$$= (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\,. \tag{6c}$$

Now, we just equate the real part of (5) with the real part of (6c), and we establish (1a). And by doing likewise with the imaginary parts, we also establish (1b).

To finish up, it's just algebra

$$\tan\left(\theta+\varphi\right) = \frac{\sin\left(\theta+\varphi\right)}{\cos\left(\theta+\varphi\right)} \tag{7a}$$

$$= \frac{\cos\theta\,\sin\varphi+\sin\theta\,\cos\varphi}{\cos\theta\,\cos\varphi-\sin\theta\,\sin\varphi} \tag{7b}$$

$$= \frac{\dfrac{\cos\theta\,\sin\varphi}{\cos\theta\,\cos\varphi}+\dfrac{\sin\theta\,\cos\varphi}{\cos\theta\,\cos\varphi}}{\dfrac{\cos\theta\,\cos\varphi}{\cos\theta\,\cos\varphi}-\dfrac{\sin\theta\,\sin\varphi}{\cos\theta\,\cos\varphi}} \tag{7c}$$

$$= \frac{\tan\varphi+\tan\theta}{1-\tan\theta\,\tan\varphi} \tag{7d}$$

$$= \frac{\tan\theta+\tan\varphi}{1-\tan\theta\,\tan\varphi}\,. \tag{7e}$$