Math Diversion Problem 470

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> What mathematicians are really interested in are
> coming up with interesting theorems and proofs.
> — Timothy Nguyen

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=hPQpYMXqj3o
Title: JEE Mains Solving Logarithm Equation with Different Bases!
 Presenter: Maths & Olympiad
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Note: JEE refers to 'Joint Entrance Exam'.

1 The Problem

Given the relation

$$\log_{20}(\sqrt[4]{x} + \sqrt{x}) = \tfrac{1}{2}\log_{16} x\,, \tag{1}$$

find the values of $x \in \mathrm{Re}^+$.

2 The Preparation

Lemma: Fundamental Rule of Logarithmic Conversion:

$$\log_a x = \frac{\log_b x}{\log_b a}\,, \tag{2}$$

where b is a positive real number.

3 The Solution

Usually, the first thing I want to do when I have a problem in two (or more) log bases is to choose one of the bases and convert all logs to that base. So, I'll convert $\log_{20}$ to $\log_{16}$, which will affect the LHS of (1). And I'll change the RHS by lifting the $\frac{1}{2}$ factor to the exponent of x. Then,

$$\frac{\log_{16}(\sqrt[4]{x}+\sqrt{x})}{\log_{16} 20} = \log_{16}\sqrt{x}\,. \tag{3}$$

Now, I'll set $c = \log_{16} 20$, then we have that

$$\log_{16}(\sqrt[4]{x}+\sqrt{x}) = c\log_{16}\sqrt{x} = \log_{16}\sqrt{x}^{\,c} = \log_{16} x^{c/2}\,. \tag{4}$$

Next, we raise 16 to the power of the last equation, to get[1]

$$\sqrt[4]{x}+\sqrt{x} = x^{c/2}\,. \tag{5}$$

Let's try an α substitution to (5). But what base should we use? It's not so obvious this time. Well, we'll eventually going to have to deal with c, which is in base 16. But $16 = 2^4$, so perhaps we should should try base 2. So, here's the next question: What is the smallest power of 2 that will clear off all the fractional exponents of x in (5)? Answer: 4. (But we still need α because we have to replace one variable with another variable.)

So, let

$$x = 2^{4\alpha}\,, \tag{6}$$

then (5) becomes

$$2^{\alpha}+2^{2\alpha} = 2^{2\alpha c}\,. \tag{7}$$

Now it's time to simplify c, or rather $2^{2\alpha c}$:

$$c = \log_{16} 20 = \frac{\log_2 20}{\log_2 16} = \tfrac{1}{4}\log_2 20\,. \tag{8}$$

Therefore,

$$2^{2\alpha c} = 2^{\alpha\frac{1}{2}\log_2 20} = 2^{\log_2 20^{\alpha/2}} = 20^{\alpha/2}\,. \tag{9}$$

So, now (7) becomes

$$2^{\alpha}+2^{2\alpha} = 20^{\alpha/2}\,. \tag{10}$$

Next, we try some small positive integer solutions in a table.

α	$2^{\alpha}+2^{2\alpha}$	$20^{\alpha/2}$
1	$2^1+2^2=5$	$20^{1/2}$
2	$2^2+2^4=20$	20^1 ✓

Table 1: An integer solution does exist!

Since $\alpha = 2$ then

$$x = 2^8 = 256\,. \tag{11}$$

[1] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\text{LHS}=\text{RHS}}$ means $b^{\text{LHS}} = b^{\text{RHS}}$.