Math Diversion 471

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> The greatest killer of creativity is interruption.
> — John Cleese

A typical induction problem.

1 The Problem

Use induction to prove **Nicomachus's Theorem:**

For any $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \,. \tag{1}$$

2 The Preparation

Let m be the smallest number on which the proposition is to be applied. (In this problem, $m = 1$, but in general it could be any integer.)

1) Show by brute calculation that $P(m)$ is true.
2) Assume $P(k)$ is true for some arbitrary $k \geq m$.
3) Show, on the basis of 1) and 2), that $P(k+1)$ is true.

Then, if one can successfully comply with all three of these requirements, then one can conclude that $P(n)$ is true for all n greater than or equal to the base case m.

Next, we let k be an arbitrary positive integer such that $P(k)$ is assumed to be true by the inductive hypothesis. (This is the one thing we get for free in this proof.) Our job now is to show, given $P(k)$ being true, that $P(k+1)$ is true.

Now, $P(k+1)$ takes the algebraic form of

$$L(k+1) = R(k+1) \,. \tag{2}$$

I intend to show this by showing that[1]

$$L(k+1) - L(k) = R(k+1) - R(k)\,. \tag{3}$$

So, why should this work? It works because it's impossible for both $L(k) = R(k)$ to be true and for (3) to be true, yet for (2) to be false.

3 The Solution

Let's begin by establishing the base case $n = 1$:

$$1^3 = (1)^2\,? \tag{4}$$

And this is true.

Then,

$$L(k+1) - L(k) = (k+1)^3\,. \tag{5}$$

And,[2]

$$R(k+1) - R(k) = \left[\frac{(k+1)(k+2)}{2}\right]^2 - \left[\frac{k(k+1)}{2}\right]^2 \cdots = (k+1)^3\,. \tag{6}$$

Therefore, we've established (3), which proves $P(k+1)$, and that completes the proof.

[1] Although the form presented in (3) may always be considered, it may not always be convenient in a proof. It's convenient in this problem because the LHS of (1) is a sum of terms. But what if, instead, it were a product of factors?

[2] I leave the details of the calculation to the interested reader.