Math Diversion Problem 472: Partial Derivatives Using SD

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1 Introduction

This is the eighth in a series of articles on how to perform the computations of partial differentiation by use of a formalism call 'Structured Differentiation'. Partial differentiation is tricky enough when we keep the set of independent variables fixed, but it can get much trickier when we allow for a change in the set of independent variables. In SD, we are permitted to refer to an independent variable as a 'fundamental' variable, and to refer to an ordered list of fundamental variables as the 'fundamental'.

2 An Example from Dynamics

We will begin with Hamiltonian dynamics. Let $F = F(t, \mathbf{q}(t), \mathbf{p}(t))$, then show that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}\,, \tag{1}$$

where $\{F, H\}$ is the Poisson bracket of F and H, and where Hamilton's equations are

$$\frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}} \tag{2a}$$

$$\frac{\partial H}{\partial \mathbf{q}} = -\dot{\mathbf{p}}\,. \tag{2b}$$

3 Solution

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\not{\partial} F}{\not{\partial} t} \tag{3}$$

$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} + \frac{\partial F}{\partial \mathbf{p}}\frac{d\mathbf{p}}{dt} \tag{4}$$

$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathbf{q}}\dot{\mathbf{q}} + \frac{\partial F}{\partial \mathbf{p}}\dot{\mathbf{p}}\,. \tag{5}$$

Using Hamilton's equations this becomes

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}} \tag{6}$$

$$= \frac{\partial F}{\partial t} + \{F, H\}\,, \tag{7}$$

where we have used the Poisson bracket of F and H. Therefore, we may write

$$\frac{\not{\partial} F}{\not{\partial} t} = \{F, H\}\,. \tag{8}$$

4 Hydrodynamics

Moving now to hydrodynamics, consider the density function

$$\rho = \rho(t, \mathbf{x}(t))\,. \tag{9}$$

Differentiating this we obtain

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\not{\partial} \rho}{\not{\partial} t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}}\dot{\mathbf{x}}\,. \tag{10}$$

Or, in the alternative notation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho\,. \tag{11}$$