Math Diversion 473

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Chance favors the prepared mind.
— Louis Pasteur

`Title: An Interesting Trigonometric Integral`
`Presenter: Patrick`

1 The Problem

Find the indefinite integral

$$I = \int \frac{dx}{\sin x} \,. \tag{1}$$

2 The Preparation

According to my table of integrals,

$$I = \int \frac{dx}{\sin x} = -\ln |\, \csc x + \cot x \,| + C \,. \tag{2}$$

3 The Solution

After having tried a lot of other ways to integrate, I have found it advantageous to make the denominator a square. But to accomplish this, I'll need to perform a virtual emplacement of an extra factor of $\sin x$ in it. Hence,

$$I = \int \frac{\sin x \, dx}{\sin^2 x} = -\int \frac{d(\cos x)}{1-\cos^2 x} = -\int \frac{dy}{1-y^2} = -\int \frac{dy}{(1-y)(1+y)} \,, \tag{3}$$

where $y = \cos x$.

At this point, I'll use the trick of partial fractions to simplify the denominator.

$$\frac{1}{(1-y)(1+y)} = \frac{A}{(1-y)} + \frac{B}{(1+y)} \,. \tag{4}$$

where A and B are real numbers. All we do now is to solve for A and B, but this is easy. We begin by multiplying through by $(1-y)(1+y)$:

$$1 = A(1+y) + B(1-y)\,. \tag{5}$$

And now the magic begins. Both sides of this equation contain both a constant part and a variable part (even if it's zero!), which will present as two coupled equations to solve algebraically:

$$1 = A + B\,, \tag{6a}$$

$$0 = A - B\,, \tag{6b}$$

where the first of these equations results from setting the constant parts of the LHS to the RHS, and the second results from setting the variable part of the LHS to the variable part of the RHS.

The solution to these two is

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}\,. \tag{7}$$

On substituting these into (4), we get

$$\frac{1}{(1-y)(1+y)} = \frac{1}{2}\frac{1}{(1-y)} + \frac{1}{2}\frac{1}{(1+y)}\,. \tag{8}$$

Therefore,

$$\begin{aligned}
I(y) &= -\int \frac{dy}{(1-y)(1+y)} && (9a)\\
&= -\frac{1}{2}\left[\int \frac{dy}{(1-y)} + \int \frac{dy}{(1+y)}\right] && (9b)\\
&= -\frac{1}{2}\left[-\int \frac{d(1-y)}{(1-y)} + \int \frac{d(1+y)}{(1+y)}\right] && (9c)\\
&= -\frac{1}{2}\left[-\ln|1-y| + \ln|1+y|\,\right] && (9d)\\
&= -\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|; && (9e)\\
I(x) &= -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C && (9f)\\
&= -\frac{1}{2}\ln\left|\frac{(1+\cos x)^2}{1-\cos^2 x}\right| + C && (9g)\\
&= -\ln\left|\frac{1+\cos x}{\sin x}\right| + C && (9h)\\
&= -\ln|\csc x + \cot x| + C\,. && (9i)
\end{aligned}$$