Math Diversion Problem 474

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Keep an open mind. That's the secret.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=WpRmwKpGNJc
Title: Solving Harvard's Exponential Problem!
 Presenter: Maths & Olympiad
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1 The Problem

Given the relation

$$4^x = (2x)^{32}\,, \tag{1}$$

find the values of x.

2 The Solution

It seems natural enough to me to convert the LHS of (1), to end up with

$$(2)^{2x} = (2x)^{32}\,, \tag{2}$$

and then to take the logarithm bass 2 across the equation:

$$2x = 32\log_2(2x) = 32(\log_2 2 + \log_2 x) = 32(1 + \log_2 x)\,. \tag{3}$$

Next, we divide through by 2:

$$x = 16(1 + \log_2 x)\,. \tag{4}$$

After attempting other ways to finish, I decided on an α substitution. So, let

$$x = 2^\alpha\,, \tag{5}$$

then (4) becomes

$$2^\alpha = 16(1 + \log_2 2^\alpha) = 16(1 + \alpha)\,. \tag{6}$$

Next, we try some small positive integer solutions in a table.

α	2^α	$16(1+\alpha)$
1	$2^1 = 2$	$16(2)$
2	$2^2 = 4$	$16(3)$
3	$2^3 = 8$	$16(4)$
4	$2^4 = 16$	$16(5)$
5	$2^5 = 32$	$16(6)$
6	$2^6 = 64$	$16(7)$
7	$2^7 = 128$	$16(8) = 128$ ✓

Table 1: An integer solution does exist!

Since $\alpha = 7$ then

$$x = 2^7 = 128\,. \tag{7}$$