Math Diversion Problem 476: Partial Derivatives Using SD

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1 Introduction

This is the ninth in a series of articles on how to perform the computations of partial differentiation by use of a formalism call 'Structured Differentiation'. Partial differentiation is tricky enough when we keep the set of independent variables fixed, but it can get much trickier when we allow for a change in the set of independent variables. In SD, we are permitted to refer to an independent variable as a 'fundamental' variable, and to refer to an ordered list of fundamental variables as the 'fundamental'.

2 An Example from Advanced Calculus

If $G_1(x_1, x_2, y)$, $G_2(x_1, x_2, y)$, and $f(x_1, x_2)$ are given, and if

$$g_i(x_1, x_2) \equiv G_i\big(x_1, x_2, f(x_1, x_2)\big) \qquad (i = 1, 2)\,, \tag{1}$$

show that

$$\left|\frac{\partial(g_1, g_2)}{\partial(x_1, x_2)}\right| = \left|\frac{\partial(G_1, G_2)}{\partial(x_1, x_2)}\right| + \frac{\partial f}{\partial x_1}\left|\frac{\partial(G_1, G_2)}{\partial(y, x_2)}\right| + \frac{\partial f}{\partial x_2}\left|\frac{\partial(G_1, G_2)}{\partial(x_1, y)}\right|\,. \tag{2}$$

First, note that g_i is the reduction of G_i to primitive form.[1] This reduction is not necessary in SD, which would have replaced the LHS of the last equation with $\left|\dfrac{\delta(G_1, G_2)}{\delta(x_1, x_2)}\right|$.

Proof: We begin by rewriting what we have in vector notation

$$\mathbf{g}(\mathbf{x}) = \mathbf{G}\big(\mathbf{x}, f(\mathbf{x})\big)\,. \tag{3}$$

[1] This means that g and G share the same independent variables.

Then,

$$\frac{\delta \mathbf{g}}{\delta \mathbf{x}} = \frac{\delta \mathbf{G}}{\delta \mathbf{x}} \,. \tag{4a}$$

Furthermore, since $\mathbf{g}$ is primitive in $\mathbf{x}$, then,

$$\frac{\delta \mathbf{g}}{\delta \mathbf{x}} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \,. \tag{4b}$$

And then

$$\begin{aligned}
\left|\frac{\partial(g_1,g_2)}{\partial(x_1,x_2)}\right| = \left|\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right| = \left|\frac{\delta \mathbf{g}}{\delta \mathbf{x}}\right| = \left|\frac{\delta \mathbf{G}}{\delta \mathbf{x}}\right| &= \left(\frac{\delta G_1}{\delta x_1}\frac{\delta G_2}{\delta x_2} - \frac{\delta G_1}{\delta x_2}\frac{\delta G_2}{\delta x_1}\right) \\
&= \left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_1}{\partial y}\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial G_2}{\partial x_2} + \frac{\partial G_2}{\partial y}\frac{\partial f}{\partial x_2}\right) \\
&\quad - \left(\frac{\partial G_1}{\partial x_2} + \frac{\partial G_1}{\partial y}\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial G_2}{\partial x_1} + \frac{\partial G_2}{\partial y}\frac{\partial f}{\partial x_1}\right) \\
&= \left(\frac{\partial G_1}{\partial x_1}\frac{\partial G_2}{\partial x_2} - \frac{\partial G_1}{\partial x_2}\frac{\partial G_2}{\partial x_1}\right) + \frac{\partial f}{\partial x_1}\left(\frac{\partial G_1}{\partial y}\frac{\partial G_2}{\partial x_2} - \frac{\partial G_1}{\partial x_2}\frac{\partial G_2}{\partial y}\right) \\
&\quad + \frac{\partial f}{\partial x_2}\left(\frac{\partial G_1}{\partial x_1}\frac{\partial G_2}{\partial y} - \frac{\partial G_1}{\partial y}\frac{\partial G_2}{\partial x_1}\right) \\
&= \left|\frac{\partial(G_1,G_2)}{\partial(x_1,x_2)}\right| + \frac{\partial f}{\partial x_1}\left|\frac{\partial(G_1,G_2)}{\partial(y,x_2)}\right| + \frac{\partial f}{\partial x_2}\left|\frac{\partial(G_1,G_2)}{\partial(x_1,y)}\right| \,.
\end{aligned}$$

Reminder: An expression like $\dfrac{\partial(g_1,g_2)}{\partial(x_1,x_2)}$ is a matrix, whereas $\left|\dfrac{\partial(g_1,g_2)}{\partial(x_1,x_2)}\right|$ is its determinant.