Math Diversion 477

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> Science is a community of scholars engaged in the production of certifiable knowledge.[1]
> — Broad & Wade

1 The Problem

Let S be a set and let $\mathrm{P}(S)$ be the so-called *power set* of S, which is the set of all subsets of S, which includes the empty set. Let A, B, C be any subsets of S. Let '$\cap$' be the symbol for *set intersection.*[2]

Show that

1) '$\cap$' is a 'binary operator' on $\mathrm{P}(S)$, meaning that for any two elements of A, B of $\mathrm{P}(S)$,

$$A \cap B \in \mathrm{P}(S)\,; \tag{1}$$

2) '$\cap$' is a commutative operator on $\mathrm{P}(S)$, meaning that for any two elements A, B of $\mathrm{P}(S)$,

$$A \cap B = B \cap A\,; \tag{2}$$

and 3) that '$\cap$' is an associative operator on $\mathrm{P}(S)$, meaning that for any three elements A, B, C of $\mathrm{P}(S)$,

$$A \cap (B \cap C) = (A \cap B) \cap C\,. \tag{3}$$

2 The Preparation

At this point, I just want to clarify some of the terms. Our first term to clarify is 'set membership', signified by '$\in$'. So, if A is a set, the element x is in the set A, if and only if $x \in A$.

Sets are by design containers of elements. A set with no elements (the 'empty set') is a legitimate set, and is designated by either '$\{\}$' or '$\emptyset$'. If A is a set,

[1] Broad, W. and N. Wade, 1982, *Betrayers of the Truth*, New York, Simon and Schuster.

[2] Try not to get confused: Each subset of S is an element of $\mathrm{P}(S)$.

any arbitrary collection of elements of A, call it W, is a *subset* of A, which is denoted by

$$W \subset A\,. \tag{4}$$

By the way, the empty set is a subset of every set.

Now, how do we decide if an element x is in the intersection of two sets?[3] Let A, B be subsets of S, then $x \in S$ is in $A \cap B$ if and only if

$$x \in A \text{ and } x \in B\,. \tag{5}$$

Finally, let's talk about **set equality**. Two sets are said to be 'equal' if every element of one set is in the other set, and vice versa. Another way to state this is that two set are equal if each is contained in the other.

3 The Solution

Part 1: This one is simple. $A \cap B$ is the collection of elements of S which are common to both A and B. Since this collection of elements of S is itself a subset of S then it is an element of $\mathrm{P}(S)$, because, by construction, every subset of S is assigned as an element of $\mathrm{P}(S)$. Hence, the intersection of any two elements of $\mathrm{P}(S)$ is itself an element of $\mathrm{P}(S)$:

$$A \cap B \in \mathrm{P}(S)\,. \tag{6}$$

Therefore, by definition, the intersection operation on the elements of $\mathrm{P}(S)$ is a binary operation on $\mathrm{P}(S)$.

So, I want to deal with a possible objection the reader may have, saying to me: It makes no sense to talk about the intersection of two elements of a set. To which, my answer is, Yes, if you pick a set at random, then it's probably meaningless to talk about the intersection of those two elements. But this situation is special. The elements of $\mathrm{P}(S)$ are themselves sets, so it's completely meaningful to talk about their intersection, especially since those sets are subsets of another set.

Part 2: '$\cap$' is a commutative operator on $\mathrm{P}(S)$, meaning that for any two elements A, B of $\mathrm{P}(S)$,

$$A \cap B = B \cap A\,. \tag{7}$$

Let's start with the LHS of (7).

$$A \cap B = \{x \in S \,|\, x \in A \text{ and } x \in B\}\,, \tag{8}$$

where the 'and' in this statement is 'logical and', and 'logical and' is commutative. Thus, using logical commutation:

$$A \cap B = \{x \in S \,|\, x \in B \text{ and } x \in A\}\,, \tag{9}$$

[3]Subsets are sets in their own right.

but this is the same set definition as for $B \cap A$, since,

$$B \cap A = \{x \in S \,|\, x \in B \text{ and } x \in A\}\,. \tag{10}$$

Thus we have shown that $A \cap B$ and $B \cap A$ contain the same elements, and thus are equal to each other. The further implication of this is that $\cap$ is a commutative operation on $\mathrm{P}(S)$.

Part 3: '$\cap$' is an associative operator on $\mathrm{P}(S)$, meaning that for any three elements A, B, C of $\mathrm{P}(S)$,

$$A \cap (B \cap C) = (A \cap B) \cap C\,. \tag{11}$$

I'm going to be a little less formal on this one. Now,

$$x \in A \cap (B \cap C) \tag{12}$$

means that

$$x \in A \text{ and } x \in (B \cap C)\,, \tag{13}$$

or

$$x \in A \text{ and } (x \in B \text{ and } x \in C)\,. \tag{14}$$

But 'logical and' is associative, hence (14) implies that

$$(x \in A \text{ and } x \in B) \text{ and } x \in C\,. \tag{15}$$

But this implies that

$$x \in A \cap (B \cap C) \implies x \in (A \cap B) \cap C\,. \tag{16}$$

And this implies that

$$A \cap (B \cap C) \subset (A \cap B) \cap C\,. \tag{17}$$

Obviously, if we ran this argument the other way, we'd end up with

$$(A \cap B) \cap C \subset A \cap (B \cap C)\,. \tag{18}$$

And these last two equations imply that

$$A \cap (B \cap C) = (A \cap B) \cap C\,. \tag{19}$$