Math Diversion 478

P. Reany

March 23, 2025

Every good mathematician is at least half a philosopher,
and every good philosopher is at least
half a mathematician.
— Gottlob Frege

```
Source: https://www.youtube.com/watch?v=DAkpGp5Ou5c
Title: Solving the Hardest Algebra Problem
 Presenter:  Maths & Olympiads
```

1 The Problem

Given the relations

$$a + b = \frac{50}{8}\,, \tag{1a}$$

$$(1 + \sqrt{a})(1 + \sqrt{b}) = \frac{15}{2}\,, \tag{1b}$$

where $a, b > 0$. Find the values of a, b.

2 The Solution

The first thing I'd like to do is to make a variable substitution:

$$x = 1 + \sqrt{a}\,, \tag{2a}$$

$$y = 1 + \sqrt{b}\,. \tag{2b}$$

Therefore,

$$a = (x - 1)^2\,, \tag{3a}$$

$$b = (y - 1)^2\,. \tag{3b}$$

So, the Given relations become,

$$(x-1)^2 + (y-1)^2 = \frac{50}{8}\,, \tag{4a}$$

$$xy = \frac{15}{2}\,. \tag{4b}$$

This means that under the change of variables, we are looking for the intersection points of a circle centered at $(1, 1)$ with a hyperbola.

To continue, let's multiply (4a) through by x^2, to get

$$x^2(x-1)^2 + (xy-x)^2 = \frac{50}{8}x^2\,. \tag{5}$$

Using (4b), this becomes

$$x^2(x-1)^2 + \left(\frac{15}{2} - x\right)^2 = \frac{50}{8}x^2\,. \tag{6}$$

On multiplying this out, we have that

$$4x^4 - 8x^3 - 17x^2 - 60x + 225 = 0\,. \tag{7}$$

I'll let WolframAlpha do this one:

$$x = 5/2, \quad 3\,. \tag{8}$$

Therefore,

$$a = (x-1)^2 = (3/2)^2 = 9/4, \quad 2^2 = 4\,. \tag{9}$$

Now, since a and b enter this problem symmtrically, we know that the two pairs of solutions are

$$(a, b) = (9/4, 4),\ (4, 9/4)\,. \tag{10}$$