Math Diversion 480

P. Reany

March 25, 2025

`Source: An induction problem in Fibonacci numbers`

1 The Problem

The Fibonacci numbers are defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \,, \tag{1}$$

where

$$F_1 = 1 \quad \text{and} \quad F_2 = 1 \,. \tag{2}$$

Use induction to show that $5 \,|\, F_{5n}$ for all $n \geq 1$.

2 The Preparation

Starting at $n = 3$, according to (1), every value for F_n is equal to the sum of its two previous terms of the sequence. So, the sequence goes as

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots\,. \tag{3}$$

As for the phrase $5 \,|\, F_{5n}$, it reads as 5 divides F_{5n}. In general, if for integers a, b, $a \,|\, b$ means that there exists an integer c such that

$$b = ac \quad \text{or} \quad b/a = c \,. \tag{4}$$

A little less technically, this means that when we divide b by a, there is no remainder.

3 The Solution

An induction proof follows this general guideline:

1) Show that the relation is true for the base case $n = 1$.

2) Assume that the relation is true for an arbitrary case $k > 1$ (the 'inductive hypothesis').

3) Show that the relation is true for the $(k+1)$st case.

---

1) For base case $n = 1$, $F_5 = 5$, therefore $5 \mid F_5$.

2) Assume that $5 \mid F_{5k}$ for $k > 1$ (the 'inductive hypothesis').

3) We need to show that $5 \mid F_{5(k+1)}$. Let's see if it does.

$$\begin{aligned} F_{5(k+1)} = F_{5k+5} &= F_{5k+4} + F_{5k+3} \\ &= (F_{5k+3} + F_{5k+2}) + (F_{5k+2} + F_{5k+1}) \\ &= F_{5k+3} + 2F_{5k+2} + F_{5k+1} \\ &= 3F_{5k+2} + 2F_{5k+1} \\ &= 5F_{5k+1} + 3F_{5k}\,. \end{aligned}$$

Clearly, 5 divides the sum of the two terms on the RHS, because it divides each term separately. Five divides the first term on the RHS because it has a factor of 5 in it, so it doesn't even matter what the value of F_{5k+1} is. And 5 divides the second term because we have applied the inductive hypothesis by assumption. Therefore $5 \mid F_{5(k+1)}$ and we are finished.