Math Diversion Problem 481: Partial Derivatives Using SD

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I am having some issues understanding what should
I keep constant and what not in certain cases
when I take partial derivatives.
— A poster to stackexchange

1 Introduction

This is the tenth in a series of articles on how to perform the computations of partial differentiation by use of a formalism call 'Structured Differentiation'. Partial differentiation is tricky enough when we keep the set of independent variables fixed, but it can get much trickier when we allow for a change in the set of independent variables. In SD, we are permitted to refer to an independent variable as a 'fundamental' variable, and to refer to an ordered list of fundamental variables as the 'fundamental'.

2 Another Example from Advanced Calculus

Given the system

$$\begin{aligned} x &= u\cos v, \\ y &= u\sin v \end{aligned} \tag{1}$$

find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.

First, we rewrite the last system as

$$\begin{aligned} u\cos v - x &= 0, \\ u\sin v - y &= 0 \end{aligned} \tag{2}$$

and then make the definitions

$$\mathbf{F} \equiv (F_1, F_2)^t = (u\cos v - x, u\sin v - y)^t \tag{3}$$

$$\mathbf{u} = (u, v)^t, \quad \mathbf{x} = (x, y)^t \tag{4}$$

Since $\mathbf{F} = \mathbf{0}$ then $\mathbf{dF}/\mathbf{dx} = \mathbf{0}$, or

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\frac{\delta \mathbf{u}}{\delta \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = 0\,. \tag{5}$$

Solving for $\partial\mathbf{u}/\partial\mathbf{x}$ we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right)^{-1}\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\,. \tag{6}$$

Now

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \begin{bmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{bmatrix} \quad \text{and} \quad \det\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right) = u\,. \tag{7}$$

So,

$$\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right)^{-1} = \frac{1}{u}\begin{bmatrix} u\cos v & u\sin v \\ -\sin v & \cos v \end{bmatrix} = \begin{bmatrix} \cos v & \sin v \\ -\dfrac{\sin v}{u} & \dfrac{\cos v}{u} \end{bmatrix} \tag{8}$$

and therefore

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} \\ &= -\begin{bmatrix} \cos v & \sin v \\ -\dfrac{\sin v}{u} & \dfrac{\cos v}{u} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos v & \sin v \\ -\dfrac{\sin v}{u} & \dfrac{\cos v}{u} \end{bmatrix}\,. \end{aligned} \tag{9}$$

Reminder: An expression like $\dfrac{\partial(g_1, g_2)}{\partial(x_1, x_2)}$ is a matrix, whereas $\left|\dfrac{\partial(g_1, g_2)}{\partial(x_1, x_2)}\right|$ is its determinant.