Math Diversion Problem 482

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> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another.
> — Keith Devlin

The problem is found at:

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Source: https://www.youtube.com/watch?v=HrcMBaccYkQ
Title: An Interesting Equation From Russia | Problem 514
 Presenter:  aplusbi
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1 The Problem

Given the relation

$$z^2 + 2\overline{z} + 1 = 0\,, \tag{1}$$

find the values of z.

2 The Solution

To continue, we will need some or all of the five identities:

$$z + \overline{z} = 2a\,, \tag{2a}$$
$$z - \overline{z} = 2ib\,, \tag{2b}$$
$$z\overline{z} = r^2\,, \tag{2c}$$
$$(z^2 + \overline{z}^2) = (z + \overline{z})^2 - 2z\overline{z} = 4a^2 - 2r^2\,. \tag{2d}$$
$$(z^2 - \overline{z}^2) = (z + \overline{z})(z - \overline{z}) = 4iab\,. \tag{2e}$$

My standard approach is to take the complex conjugate of (1) and then solve that system.

$$\overline{z}^2 + 2z + 1 = 0\,, \tag{3}$$

By adding this last equation to (1), we get

$$(z^2 + \overline{z}^2) + 2(z + \overline{z}) + 2 = 0\,, \tag{4}$$

which simplifies down to

$$4a^2 - 2r^2 + 4a + 2 = 0\,. \tag{5}$$

Hmmm. Let's return to the system, but this time subtract them.

$$(z^2 - \overline{z}^2) + 2(\overline{z} - \overline{z}) = 0\,, \tag{6}$$

or

$$4iab - 4ib = 0\,. \tag{7}$$

After some more simplifying, we have that

$$ab - b = 0\,. \tag{8}$$

Case 1) $b = 0$. Putting this value for b into (5), we get $a = -1$. This gives us the root for z:

$$z = -1\,. \tag{9}$$

Case 2) For $b \neq 0$ then $a = 1$. Putting this value for a into (5), we get $b = \pm 2$. This gives us the additional roots for z:

$$z = 1 \pm 2i\,. \tag{10}$$