Math Diversion 484

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> The ignorant always loud in argument.
> — Charlie Chan

A typical induction problem.

1 The Problem

Use induction to prove this result.

For any $n \geq 1$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\,. \tag{1}$$

2 The Preparation

Let m be the smallest number on which the proposition is to be applied. (In this problem, $m = 1$, but in general it could be any integer.)

1) Show by brute calculation that $P(m)$ is true.
2) Assume $P(k)$ is true for some arbitrary $k \geq m$.
3) Show, on the basis of 1) and 2), that $P(k+1)$ is true.

Then, if one can successfully comply with all three of these requirements, then one can conclude that $P(n)$ is true for all n greater than or equal to the base case m.

Next, we let k be an arbitrary positive integer such that $P(k)$ is assumed to be true by the inductive hypothesis. (This is the one thing we get for free in this proof.) Our job now is to show, given $P(k)$ being true, that $P(k+1)$ is true.

Now, $P(k+1)$ takes the algebraic form of

$$L(k+1) = R(k+1)\,. \tag{2}$$

I intend to show this by showing that[1]

$$L(k+1) - L(k) = R(k+1) - R(k)\,. \tag{3}$$

So, why should this work? It works because it's impossible for both $L(k) = R(k)$ to be true and for (3) to be true, yet for (2) to be false.

3 The Solution

Let's begin by establishing the base case $n = 1$: Is

$$1^2 = \frac{1(2)(3)}{6}\,? \tag{4}$$

Yes, this is true.

Then, for the differences on the left-hand side:

$$L(k+1) - L(k) = \sum_{i=1}^{k+1} i^2 - \sum_{i=1}^{k} i^2 = (k+1)^2\,. \tag{5}$$

And, for the differences on the right-hand side

$$\begin{aligned} R(k+1) - R(k) &= \frac{(k+1)(k+2)(2k+3)}{6} - \frac{k(k+1)(2k+1)}{6} \\ &= \frac{k+1}{6}\left[(k+2)(2k+1+2) - k(2k+1)\right] \\ &= \frac{k+1}{6}\left[(k+2)(2k+1) + 2(k+2) - k(2k+1)\right] \\ &= \frac{k+1}{6}\left[2(2k+1) + 2(k+2)\right] \\ &= \frac{k+1}{6}\left[6k+6\right] \\ &= (k+1)^2\,. \end{aligned} \tag{6}$$

Therefore, we've established (3), which proves $P(k+1)$, and that completes the proof.

[1] Although the form presented in (3) may always be considered, it may not always be convenient in a proof. It's convenient in this problem because the LHS of (1) is a sum of terms. But what if, instead, it were a product of factors?