Math Diversion Problem 487

P. Reany

March 29, 2025

> All my life I kept running into smart people.... In school there were lots of smarter kids. And when I first joined the force, sir, they had some very clever people there. And I could tell right away that it wasn't going to be easy making detective as long as they were around. What I figured that... if I worked harder than they did. Put in more time. Read the books. Kept my eyes open. Maybe I could make it happen. And I did!
>
> — Lt. Columbo to his prisoner (from the TV show *Columbo*) ("The Bye Bye Sky High IQ Murder Case")

The YouTube video is found at:

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Source: ???
Title: ???
 Presenter: ???
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1 The Problem

Given the relation

$$x^{x^{20}} = 256\,, \tag{1}$$

find the real values of x.

2 The Solution

I'll start off with a standard α transformation:

$$x = 2^{\alpha}\,. \tag{2}$$

Then (1) becomes

$$2^{\alpha 2^{20\alpha}} = 256 = 2^8\,, \tag{3}$$

On equating exponents, we get

$$\alpha 2^{20\alpha} = 8 = 2^3\,, \tag{4}$$

Okay, so I tried to finish this with an α table, without success. So, I decided to take the logarithm base 2 across this last equation, which left me with

$$\log_2 \alpha + 20\alpha = 3\,. \tag{5}$$

I didn't try to solve this with an α table, though that might have worked. Instead, I made a second substitution:

$$\alpha = 2^\beta\,. \tag{6}$$

Then (5) becomes

$$\beta + 20 \cdot 2^\beta = 3\,. \tag{7}$$

So, time to make a 'β' table.

β	$\beta + 20 \cdot 2^\beta$	3
1	$1 + 20 \cdot 2 = 41$	
-1	$-1 + 20 \cdot 2^{-1} = 9$	
-2	$-2 + 20 \cdot 2^{-2} = 3$	✓

Table 1: An integer solution does exist!

Thus,

$$\alpha = 2^{-2} = \frac{1}{4}\,, \tag{8}$$

and thus

$$x = 2^\alpha = \sqrt[4]{2}\,. \tag{9}$$

However, we must apply due diligence to rule in or out the possible solution $x = -\sqrt[4]{2}$, which does work. Hence, for real x,

$$x = \pm\sqrt[4]{2}\,. \tag{10}$$