Math Diversion 490

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Suspicion often father of truth.
— Charlie Chan

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Source: https://www.youtube.com/watch?v=OLRqI7ICN50
Title: A Deceivingly Difficult Differential Equation
 Presenter: SyberMath
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1 The Problem

Find the solution to the differential equation

$$y'' = 2y^3\,. \tag{1}$$

2 The Solution

I'd like to start by buying a y' to slap onto both sides.

$$y'y'' = 2y^3y'\,. \tag{2}$$

Now we can partially integrate this (half a first integration):

$$\frac{d}{dx}\tfrac{1}{2}(y')^2 = \tfrac{1}{2}\frac{d}{dx}y^4\,. \tag{3}$$

Then we finish the first integration, to get,

$$(y')^2 = y^4 + c_1\,. \tag{4}$$

Then take the square root.

$$y' = \pm\sqrt{y^4 + c_1}\,. \tag{5}$$

Next, the separation of variables and then integrate.

$$\int \frac{dy}{\sqrt{y^4 + c_1}} = \pm \int dx = \pm x + c_2\,. \tag{6}$$

Now, when I put this into WolframAlpha, I got out a really complicated function of y, which I could never invert to solve for x.

So, to make things a bit easier, let me assume that the initial conditions will set $c_1 = 0$ in (4), leaving

$$(y')^2 = y^4 \,. \tag{7}$$

Then (6) will become

$$\int \frac{dy}{y^2} = \pm \int dx = \pm x - c \,. \tag{8}$$

This integrates to

$$\frac{1}{y} = \pm x + c \,. \tag{9}$$

Thus,

$$y = \frac{1}{\pm x + c} \,. \tag{10}$$

Now, let's test it.

$$y' = -\frac{\pm 1}{(\pm x + c)^2} \,. \tag{11}$$

And

$$y'' = 2\frac{1}{(\pm x + c)^3} = 2y^3 \,. \tag{12}$$

Okay.