# Math Diversion 491

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April 1, 2025

In mathematics, you don't understand things.
You just get used to them.
— John von Neumann

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Source: https://www.youtube.com/watch?v=iINFoFjhtr8
Title: Math Olympiad | Solve for x + y |
 Presenter: VIJAY Maths
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## 1 The Problem

Given the relations

$$(x^3 + y^3)^3 = x^9 + y^9 \,, \tag{1a}$$

$$xy = -2 \,, \tag{1b}$$

find the real values of $x + y$.

## 2 The Preparation

Pascal's Triangle gives the coefficients to exand $(a + b)^3$:

1 3 3 1

## 3 The Solution

Expanding the LHS of (1a), we have that

$$(x^3)^3 + 3(x^3)^2(y^3) + 3(x^3)(y^3)^2 + (y^3)^3 = x^9 + y^9 \,. \tag{2}$$

On canceling out the obvious terms, we get

$$3(x^3)^2(y^3) + 3(x^3)(y^3)^2 = 0 \,. \tag{3}$$

After factoring, we have

$$3x^3y^3[x^3 + y^3] = 0 \,. \tag{4}$$

Because of the condition (1b), neither $x$ nor $y$ can be zero, hence

$$x^3 + y^3 = 0\,. \tag{5}$$

Since we've been asked to solve for $x + y$, let's use this as a hint:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 3x^2y + 3xy^2\,, \tag{6}$$

where we used (5). Then

$$(x+y)^3 = 3xy(x+y)\,. \tag{7}$$

Now, since (1b) tells us that $xy \neq 0$, then a real solution to $x + y$ is

$$x + y = 0\,. \tag{8}$$

But if $x + y \neq 0$ then (7) can be simplified down to

$$(x+y)^2 = 3xy\,. \tag{9}$$

Now we again use (1b), and this becomes

$$(x+y)^2 = -6\,. \tag{10}$$

On taking the square root, we have that

$$x + y = \pm i\sqrt{6}\,. \tag{11}$$