Math Diversion Problem 496

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Our greatest weakness lies in giving up. The most
certain way to succeed is always to try
just one more time.
—Thomas Edison

The problem is found at:

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Source: https://www.youtube.com/watch?v=wLbcOXdqESU
Title: An Imaginary Cubic Equation | Problem 509
 Presenter:  aplusbi
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1 The Problem

Given the relation

$$z^3 + 2z = i\,, \tag{1}$$

find the values of z.

2 The Preparation

To continue, we will need some or all of the seven identities:

$$z + \overline{z} = 2a\,, \tag{2a}$$
$$z - \overline{z} = 2ib\,, \tag{2b}$$
$$z\overline{z} = r^2\,, \tag{2c}$$
$$z^2 + \overline{z}^2 = (z + \overline{z})^2 - 2z\overline{z} = 4a^2 - 2r^2\,. \tag{2d}$$
$$z^2 - \overline{z}^2 = (z + \overline{z})(z - \overline{z}) = 4iab\,. \tag{2e}$$
$$z^3 + \overline{z}^3 = 8a^3 - 6ar^2\,. \tag{2f}$$
$$z^3 - \overline{z}^3 = 2bi(4a^2 - r^2)\,. \tag{2g}$$

3 The Solution

Having tried my usual method of taking the complex conjugate of (1), to get

$$\overline{z}^3 + 2\overline{z} = -i\,, \tag{3}$$

and deciding that the resulting equations in cubics and quadratics in variables a, b, and r is just too difficult to solve by hand, I decided to look for a quick solution: in this case by trying the obvious choice of $z = bi$, since in (1), z is always to an odd power, then the i's will cancel out.

So, substituting $z = bi$ in (1), we get

$$(bi)^3 + 2(bi) = i\,, \tag{4}$$

which becomes

$$-ib^3 + 2ib = i\,, \tag{5}$$

which becomes

$$b^3 - 2b + 1 = 0\,, \tag{6}$$

which has the obvious root

$$b = 1\,. \tag{7}$$

Therefore, we found one root

$$z = i\,. \tag{8}$$

But there's still two more roots left in (6) from a quadratic factor. The way to get this factor is to divide $b^3 - 2b + 1$ by $b - 1$, to get a result to set equal to zero.

$$b^2 + b - 1 = 0\,. \tag{9}$$

By the quadratic formula, we get

$$b = \tfrac{1}{2}(-1 \pm \sqrt{5})\,, \tag{10}$$

which implies for z:

$$z = \frac{i}{2}(-1 \pm \sqrt{5})\,, \tag{11}$$