Math Diversion Problem 497

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No mystery is closed to an open mind.
— Tim White
TV show *Sightings*

The problem is found at:

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Source: https://www.youtube.com/watch?v=33wTO_Lo7wA
Title: What is mathematical elegance?
 Presenter:  Mathemaniac
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1 The Problem

Given are two adjacent squares — the smaller one of side length 4 and the larger one of side length x. Find the area of triangle ABC.

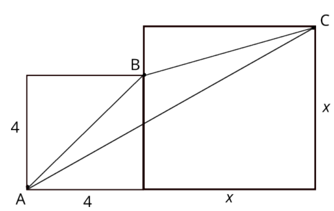

Figure 1. Two adjacent squares, one of side length 4, the other of side length x. What is the area of triangle ABC?

Mathemaniac solved this problem in two different ways. I will solve it in a third way, using **geometric algebra**.

2 The Preparation: Geometric Algebra

Geometric Algebra is an enormous subject these days, containing and generalizing such disciplines as complex numbers, vector algebra in n dimensions, tensors, Clifford Algebra, and so on. But all we will need to solve this problem is a bare minimum of it.

All we really need is the vector algebra of the plane. Vectors can be added, subtracted, and scalar multiplied. Beyond that we can take the inner (or dot) product of two vectors **a** and **b**, which is the same as in Gibbs's vector algebra that most people are familiar with, namely $\mathbf{a} \cdot \mathbf{b}$, which is a scalar. Two nonzero vectors are orthogonal if and only if their dot product is zero.

One of the convenient generalization that the geometric algebra has over Gibbs's vector algebra is the **wedge** product, which is used to form oriented area sections of a plane (also called a **bivector**). This product is similar to the cross product of ordinary vector algebra. For instance, they can both be used to calculate the area of a parallelogram by taking their absolute values:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a} \wedge \mathbf{b}| = \text{area}\,. \tag{1}$$

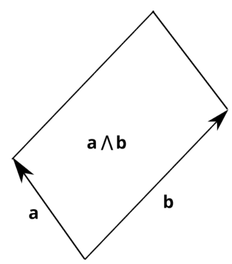

Figure 2. The wedge of two vectors forms a bivector.

The wedge product is antisymmetric, so

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}\,. \tag{2}$$

On connecting two opposite corners of a parallelogram with a line segment, we divide the parallelogram into two equal parts, each a triangle.

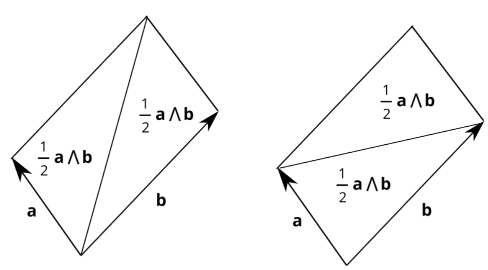

Figure 3. The line segment divides the parallelogram into two equal-area triangles. I'm ignoring the orientation of the bivectors.

If vectors $\mathbf{s}$ and $\mathbf{t}$ lie on the same line, they will not generate, by being wedged together, a nonzero area bivector: Hence, $\mathbf{s} \neq 0$ and $\mathbf{t} \neq 0$ will lie on the same line, if and only if they are scalar multiples of each other, and thus

$$\mathbf{s} \wedge \mathbf{t} = 0\,. \tag{3}$$

Let $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ any two mutually orthogonal unit vectors of a given plane such that $\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2$ has positive orientation in the plane.[1] These vectors can be used to setup a sort of 'coordinate system' for the plane.

3 The Solution

In Fig. 4, we can see the addition of some vectors to the original diagram, the two $\boldsymbol{\sigma}$'s represent a pair of orthonormal vectors in the plane, by which all other vectors can be described.

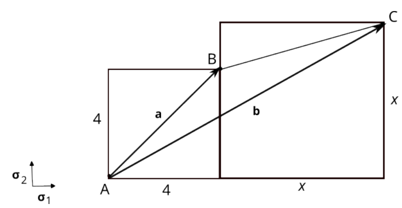

Figure 4. Vectors added to the original diagram.

In terms of the given information, vectors $\mathbf{a}$ and $\mathbf{b}$ can be represented as

$$\mathbf{a} = 4\boldsymbol{\sigma}_1 + 4\boldsymbol{\sigma}_2 = 4(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,, \tag{4a}$$
$$\mathbf{b} = (4+x)\boldsymbol{\sigma}_1 + x\boldsymbol{\sigma}_2 = 4\boldsymbol{\sigma}_1 + x(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,. \tag{4b}$$

We can now calculate $\mathbf{a} \wedge \mathbf{b}$:

$$\mathbf{a} \wedge \mathbf{b} = [4(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)] \wedge [4\boldsymbol{\sigma}_1 + x(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)] \tag{5a}$$
$$= [4(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)] \wedge [4\boldsymbol{\sigma}_1] \tag{5b}$$
$$= 16\,\boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1\,. \tag{5c}$$

So, the area of the triangle ABC is given as

$$\tfrac{1}{2}\,|\mathbf{a} \wedge \mathbf{b}| = \tfrac{1}{2}(16\,|\boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1|) = \tfrac{1}{2}16 = 8\,, \tag{6}$$

where we used that $|\boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1| = 1$.

It's interesting that the area is not a function of x.

[1] This means that if their tails are pinned together, $\boldsymbol{\sigma}_1$ is rotated into $\boldsymbol{\sigma}_2$ in a anticlockwise manner.