Math Diversion Problem 498: Jacobians in Advanced Calculus

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> Matters of notation play a considerable role in connection with the chain rule. Wide varieties of usage exist in mathematical writing where the chain rule is concerned.
>
> —Taylor & Mann

Source: Advanced Calculus (Taylor & Mann)
Title: Jacobians in Advanced Calculus
Presenter: Patrick

1 Problems

PROBLEM:
Given a surface defined by $F(x, y, z) = 0$, we know that if $\partial F/\partial z \neq 0$ then, by the Implicit Function Theorem, $z = z(x, y)$. Show that the direction of the normal to the surface is given by the direction ratios

$$\frac{\partial z}{\partial x} \quad : \quad \frac{\partial z}{\partial y} \quad : \quad -1\,. \tag{1}$$

SOLUTION:
From vector calculus we know that at any point on the surface defined by the locus of points $F(x, y, z) = 0$ where the normal is defined, the direction of the normal to the surface is given by the gradient $\partial F/\partial \mathbf{x}$. Now, note that since $F = 0$, then

$$\frac{\delta F}{\delta x} = \frac{\partial F}{\partial x} + \frac{\not\partial F}{\not\partial x} = 0\,, \quad \frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} + \frac{\not\partial F}{\not\partial y} = 0 \tag{2}$$

from which we conclude that

$$\frac{\partial F}{\partial x} = -\frac{\not\partial F}{\not\partial x}\,, \quad \frac{\partial F}{\partial y} = -\frac{\not\partial F}{\not\partial y} \tag{3}$$

Thus,

$$\begin{aligned}
\frac{\partial F}{\partial \mathbf{x}} &= \left(\frac{\partial F}{\partial x}\,, \quad \frac{\partial F}{\partial y}\,, \quad \frac{\partial F}{\partial z}\right) \\
&= \left(-\frac{\not\partial F}{\not\partial x}\,, \quad -\frac{\not\partial F}{\not\partial y}\,, \quad \frac{\partial F}{\partial z}\right) \\
&= -\left(\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\,, \quad \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\,, \quad -\frac{\partial F}{\partial z}\right) \\
&= -\frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\,, \quad \frac{\partial z}{\partial y}\,, \quad -1\right) \qquad (4)
\end{aligned}$$

Therefore the direction ratios of the normal to the surface is given by (1).

PROBLEM:

Given a surface defined by

$$\begin{cases} x = x \\ y = y \\ z = z(x, y) \end{cases} \tag{5}$$

the direction of the normal to the surface is given by the direction ratios

$$\frac{\partial z}{\partial x} \quad : \quad \frac{\partial z}{\partial y} \quad : \quad -1\,. \tag{6}$$

Now, note that a surface in 3-space needs only two intrinsic variables to specify coordinates on it. Thus we can then rewrite the coordinates of the surface in the 3d embedding space as

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \tag{7}$$

Show that the direction ratios of the normal can be given by

$$j_1 \quad : \quad j_2 \quad : \quad j_3 \tag{8}$$

where

$$j_1 = \left|\frac{\partial(y, z)}{\partial(u, v)}\right| \quad : \quad j_2 = \left|\frac{\partial(z, x)}{\partial(u, v)}\right| \quad : \quad j_3 = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\,. \tag{9}$$

SOLUTION:

As always in this type of problem, the solution is most easily perceived only after make the standard simplifications. One of these is to write the "state" vector, $\mathbf{x} = (x, y, z)^t$. Another is to determine the before and after fundamentals: The before fundamental is $\boldsymbol{\eta} = (x, y)^t$ and the after is $\boldsymbol{\eta}' = (u, v)^t$. So, we can write

$$\mathbf{x}(\boldsymbol{\eta}) = \mathbf{x}(\boldsymbol{\eta}'(\boldsymbol{\eta}))\,. \tag{10}$$

Note that the two variables x, y are convoluted through $\boldsymbol{\eta}'$ to $\boldsymbol{\eta}$. The jacobian J_T of the transformation

$$\boldsymbol{\eta}' = T(\boldsymbol{\eta}) = \boldsymbol{\eta}'(\boldsymbol{\eta}) \tag{11}$$

is then

$$J_T = \left| \frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\eta}} \right| = j_3^{-1}\,. \tag{12}$$

Differentiating (10) and simplifying the total derivatives to partials we get

$$\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}'} \frac{\partial \boldsymbol{\eta}'}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}'}\, \mathbf{j}_3^{-1} \tag{13}$$

On writing this last equation in matrix form and simplifying we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1} \right). \tag{14}$$

Now, we need to get determinant-form information from this, but how do we do this when we don't have square matrices? Actually, the square matrices are in (14) – there are three of them – and we can solve the problem from here by inspection. But let's prove that this is legal. Notice that the result of multiplying the 3×2 matrix by the 2×2 matrix on the right of (14) (to produce another 3×2 matrix) does not intermix the elements of the rows of the original 3×2 matrix. In other words, the result of the multiplication of the two matrices produces a matrix whose any two rows could be calculated without the presence of the third row — even in the original 3×2 matrix. So, we can decompose (14) into three equations of 2×2 matrices from which we can take determinants.

Let's do that now. Taking the first two rows we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1} \right). \tag{15}$$

From rows 1 and 3 we get

$$\begin{pmatrix} 1 & 0 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1} \right). \tag{16}$$

And, finally, from rows 2 and 3 we get

$$\begin{pmatrix} 0 & 1 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right). \tag{17}$$

From the determinant of (15) we get

$$1 = j_3\, j_3^{-1}\,. \tag{18}$$

From the determinant of (16) we get

$$\frac{\partial z}{\partial y} = -j_2 j_3^{-1}\,. \tag{19}$$

From the determinant of (17) we get

$$\frac{\partial z}{\partial x} = -j_1 j_3^{-1}\,. \tag{20}$$

So, on substituting these values into (6) we get

$$-j_1/j_3 \quad : \quad -j_2/j_3 \quad : \quad -1 \tag{21}$$

which is equivalent to

$$j_1 \quad : \quad j_2 \quad : \quad j_3\,. \tag{22}$$

2 Afterwards

The approach I took in this paper, and in so many of my other papers on this topic, is not the only way to approach doing these transformations and their derivatives. But this method is orderly, predictable, and very often works.