Math Diversion Problem 500

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> The human mind has never invented a
> labor-saving machine equal to algebra.
> — J. Willard Gibbs

The problem is found at:

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Title: A Trig Identity
 Presenter:  Patrick
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1 The Problem

By use of complex numbers, establish the identity

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\,. \tag{1}$$

The Preparation

Euler's Formula will help us out here:

$$e^{i\theta} = \cos\theta + i\sin\theta\,. \tag{2}$$

A few lines of Pascal's Triangle.

$n=0$:						1							
$n=1$:					1		1						
$n=2$:				1		2		1					
$n=3$:			1		3		3		1				
$n=4$:		1		4		6		4		1			
$n=5$:	1		5		10		10		5		1		
$n=6$:	1		6		15		20		15		6		1

[borrowed from https://www.bedroomlan.org/coding/pascals-triangle-latex]

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2 The Solution

Let's follow the generous hint above and take the fifth power of Euler's Formula:

$$e^{5i\theta} = [\,\cos\theta + i\sin\theta\,]^5\,. \tag{3}$$

We can use Euler's formula in one other way. We can replace the θ in (2) by 5θ, to get

$$e^{5i\theta} = \cos 5\theta + i\sin 5\theta\,. \tag{4}$$

Now we can set the RHSs of these last two equations equal to each other, to get

$$\cos 5\theta + i\sin 5\theta = [\,\cos\theta + i\sin\theta\,]^5\,. \tag{5}$$

So, what we've been asked to show will require us to use only the real parts of both sides of this last equation, as they must be equal. The real part on the LHS is, of course, $\cos 5\theta$. The real part of the RHS starts with $\cos^5\theta$ and includes every other term.

Why? Because those are the only terms in which $(i\sin\theta)$ will be taken to an even power in the expansion, including the zeroth power. As a result, those terms will not contain an unpaired i, making them real valued. The other terms can be ignored as they will contain an unpaired i, making them imaginary. And let's note that $i^2 = -1$ and $i^4 = 1$.

Okay, according to Pascal's Triangle, we need the coefficients for the expansion:

$$1 \qquad\qquad 10 \qquad\qquad 5 \tag{6}$$

Therefore,

$$\cos 5\theta = (1)(\cos^5\theta)\,(i\sin\theta)^0 + (10)(\cos^3\theta)\,(i\sin\theta)^2 + (5)(\cos\theta)(i\sin\theta)^4 \tag{7a}$$

$$= \cos^5\theta - 10(\cos^3\theta)\,(\sin^2\theta) + 5(\cos\theta)\sin^4\theta \tag{7b}$$

$$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2 \tag{7c}$$

$$= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta\,(1 - 2\cos^2\theta + \cos^4\theta) \tag{7d}$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\,. \tag{7e}$$