Math Diversion Problem 503

P. Reany

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The Axiom of Choice is obviously true; the Well Ordering
Principle is obviously false; and who can tell
about Zorn's Lemma?
— Jerry Bona

1 The Problem

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Source: Abstract Algebra
Title: A group theory problem
Presenter:  Patrick
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Let

$$G = \mathrm{GL}_2(\mathbb{R}) \tag{1}$$

be the set of all 2×2 matrices with real entries and nonvanishing determinants. ('GL' stands for *general linear*'.)[1] Show that G is a group. The binary operation here is matrix multiplication. (A basic knowledge of matrices is assumed.)

2 Preparation

What is a **group**? A **group** is a nonempty set, say G, whose elements satisfy the following requirements.

1) There exists a binary operation '$*$' on the elements of G such that for any two elements a, b of G, $a * b \in G$. (Commonly used replacements for '$*$' are '$+$', '$\cdot$', or even just juxtaposition.)

3) The binary operation on G is associative. This means that for any three elements a, b, c of G,

$$(a * b) * c = a * (b * c)\,. \tag{2}$$

4) Every element a of G has an inverse, usually expressed either $-a$ or

[1] General linear is the most inclusive collection of $n \times n$ matrices, each having an inverse.

a^{-1}. Therefore $aa^{-1} = e$ for the multiplicative case, where we used juxtaposition.

Lemma: Let A, B be any two $n \times n$ matrices. Then

$$\det(AB) = \det A \ \det B \,. \tag{3}$$

3 The Solution

1) The binary operation defined on G is matrix multiplication. This is a good start towards proving that the binary operation is closed, because the product of two $n \times n$ matrices is an $n \times n$ matrix. But we also have to show that the product of any two matrices in G, say A, B, each of which has nonzero determinant, also has nonzero determinant. From the above lemma:

$$\det(AB) = \det A \ \det B \,. \tag{4}$$

Now, since neither $\det A$ nor $\det B$ is zero, then their product also is nonzero. Thus, $\det(AB)$ is nonzero. Done.

2) The set G defined in (1) certainly is nonempty, and has the identity element

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5}$$

3) The binary operation on G is associative? Yes. It's well known that matrix multiplication is associative, and I'll leave it at that.

4) Show that every element of G, say A, has an inverse. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{6}$$

There is a formula for A^{-1}:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} , \tag{7}$$

where

$$\det A = ad - bc \,. \tag{8}$$

The first thing to note about this inverse is that since $\det A$ is nonzero, then the inverse of A is well-defined (no infinities). So, what about the $\det(A^{-1})$? We must show that it is nonzero.

$$\det(A^{-1}) = \frac{1}{(\det A)^2} \det \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{(\det A)^2} (da - (-c)(-b)) = \frac{1}{\det A} \neq 0 \,. \tag{9}$$

4 Remarks

I skipped a step in that last line (9), so I'll address that now. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad B = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda A \,, \tag{10}$$

where λ is just a real number. Then

$$B = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} , \tag{11}$$

which demonstrates how to multiply a matrix by a scalar. Then

$$\begin{aligned} \det B = \det \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \\ &= \lambda^2 ad - \lambda^2 cb = \lambda^2 (ad - cb) = \lambda^2 \det A \,. \end{aligned} \tag{12}$$

Now, if $\lambda \neq 0$ and $\det A \neq 0$ then $\det B \neq 0$.